Chapter 3

Linear Modulation Techniques

Contents

3.1 Linear Modulation .................................................. 3-3
  3.1.1 Double-Sideband Modulation (DSB) ............. 3-3
  3.1.2 Amplitude Modulation ................................. 3-8
  3.1.3 Single-Sideband Modulation ....................... 3-21
  3.1.4 Vestigial-Sideband Modulation ............. 3-35
  3.1.5 Frequency Translation and Mixing .......... 3-38
3.2 Interference ....................................................... 3-46
  3.2.1 Interference in Linear Modulation .......... 3-46
3.3 Sampling Theory ............................................... 3-49
3.4 Analog Pulse Modulation .................................. 3-54
  3.4.1 Pulse-Amplitude Modulation (PAM) ......... 3-54
  3.4.2 Pulse-Width Modulation (PWM) ............ 3-56
  3.4.3 Pulse-Position Modulation .................... 3-56
3.5 Delta Modulation and PCM ............................... 3-57
  3.5.1 Delta Modulation (DM) ....................... 3-57
  3.5.2 Pulse-Code Modulation (PCM) ............ 3-60
3.6 Multiplexing ..................................................... 3-63
3.6.1 Time-Division Multiplexing (TDM) . . . . 3-64
• We are typically interested in locating a message signal to some new frequency location, where it can be efficiently transmitted.

• The carrier of the message signal is usually sinusoidal.

• A modulated carrier can be represented as

\[ x_c(t) = A(t) \cos \left[ 2\pi f_c t + \phi(t) \right] \]

where \( A(t) \) is linear modulation, \( f_c \) the carrier frequency, and \( \phi(t) \) is phase modulation.

3.1 Linear Modulation

• For linear modulation schemes, we may set \( \phi(t) = 0 \) without loss of generality.

\[ x_c(t) = A(t) \cos(2\pi f_c t) \]

with \( A(t) \) placed in one-to-one correspondence with the message signal.

3.1.1 Double-Sideband Modulation (DSB)

• Let \( A(t) \propto m(t) \), the message signal, thus

\[ x_c(t) = A_c m(t) \cos(2\pi f_c t) \]

• From the modulation theorem it follows that

\[ X_c(f) = \frac{1}{2} A_c M(f - f_c) + \frac{1}{2} A_c M(f + f_c) \]
DSB time domain waveforms

DSB spectra

**Coherent Demodulation**

- The received signal is multiplied by the signal \( 2 \cos(2\pi f_c t) \), which is synchronous with the transmitter carrier.

\[
\begin{align*}
    m(t) & \quad \rightarrow \quad x_c(t) \\
    A_c \cos[2\pi f_c t] & \quad \rightarrow \quad x_r(t) \\
    2\cos[2\pi f_c t] & \quad \rightarrow \quad d(t) \\
    y_D(t) & \quad \rightarrow 
\end{align*}
\]
3.1. LINEAR MODULATION

- For an ideal channel \( x_r(t) = x_c(t) \), so

\[
d(t) = \left[A_c m(t) \cos(2\pi f_c t) \right] 2 \cos(2\pi f_c t) \\
= A_c m(t) + A_c m(t) \cos(2\pi (2f_c) t)
\]

where we have used the trig identity \( 2 \cos^2 x = 1 + \cos 2x \)

- The waveform and spectra of \( d(t) \) is shown below (assuming \( m(t) \) has a triangular spectrum in \( D(f) \))

![Waveform and spectrum of \( d(t) \)]

- Typically the carrier frequency is much greater than the message bandwidth \( W \), so \( m(t) \) can be recovered via lowpass filtering

- The scale factor \( A_c \) can be dealt with in downstream signal processing, e.g., an automatic gain control (AGC) amplifier
• Assuming an ideal lowpass filter, the only requirement is that the cutoff frequency be greater than $W$ and less than $2f_c - W$

• The difficulty with this demodulator is the need for a coherent carrier reference

• To see how critical this is to demodulation of $m(t)$ suppose that the reference signal is of the form

$$c(t) = 2\cos[2\pi f_c t + \theta(t)]$$

where $\theta(t)$ is a time-varying phase error

• With the imperfect carrier reference signal

$$d(t) = A_c m(t) \cos \theta(t) + A_c m(t) \cos[2\pi f_c t + \theta(t)]$$

$$y_D(t) = m(t) \cos \theta(t)$$

• Suppose that $\theta(t)$ is a constant or slowly varying, then the $\cos \theta(t)$ appears as a fixed or time varying attenuation factor

• Even a slowly varying attenuation can be very detrimental from a distortion standpoint

  – If say $\theta(t) = \Delta f t$ and $m(t) = \cos(2\pi f_m t)$, then

$$y_D(t) = \frac{1}{2} [\cos[2\pi (f_m - \Delta f) t] + \cos[2\pi (f_m + \Delta f) t]]$$

  which is the sum of two tones

• Being able to generate a coherent local reference is also a practical manner
• One scheme is to simply square the received DSB signal

\[
x_r^2(t) = A_c^2 m^2(t) \cos^2(2\pi f_c t)
\]

\[
= \frac{1}{2} A_c^2 m^2(t) + \frac{1}{2} A_c^2 m^2(t) \cos[2\pi (2f_c) t]
\]

Carrier recovery concept using signal squaring

• Assuming that \( m^2(t) \) has a nonzero DC value, then the double frequency term will have a spectral line at \( 2f_c \) which can be divided by two following filtering by a narrowband bandpass filter, i.e., \( \mathcal{F}\{m^2(t)\} = k\delta(f) + \cdots \)

Note that unless \( m(t) \) has a DC component, \( X_c(f) \) will not contain a carrier term (read \( \delta(f \pm f_c) \)), thus DSB is also called a suppressed carrier scheme
Consider transmitting a small amount of unmodulated carrier

\[ m(t) \quad \rightarrow \quad \times \quad \rightarrow \quad y(t) \quad \rightarrow \quad x_{c}(t) \]

\[ A_{c}\cos 2\pi f_{c} t \quad \rightarrow \quad A_{c} M(0)/2 \]

use a narrowband filter (phase-locked loop) to extract the carrier in the demod.

\[ k \ll 1 \]

3.1.2 Amplitude Modulation

Amplitude modulation (AM) can be created by simply adding a DC bias to the message signal

\[ x_{c}(t) = [A + m(t)] A'_{c} \cos(2\pi f_{c} t) \]
\[ = A_{c} [1 + am_{n}(t)] \cos(2\pi f_{c} t) \]

where \( A_{c} = AA'_{c} \), \( m_{n}(t) \) is the normalized message such that \( \min m_{n}(t) = -1 \),

\[ m_{n}(t) = \frac{m(t)}{\min m(t)} \]

and \( a \) is the modulation index

\[ a = \frac{|\min m(t)|}{A} \]
3.1. LINEAR MODULATION

Generation of AM and a sample waveform

- Note that if \( m(t) \) is symmetrical about zero and we define \( d_1 \) as the peak-to-peak value of \( x_c(t) \) and \( d_2 \) as the valley-to-valley value of \( x_c(t) \), it follows that

\[
a = \frac{d_1 - d_2}{d_1 + d_2}
\]

**proof:** \( \max m(t) = -\min m(t) = |\min m(t)| \), so

\[
\frac{d_1 - d_2}{d_1 + d_2} = \frac{2[(A + |\min m(t)|) - (A - |\min m(t)|)]}{2[(A + |\min m(t)|) + (A - |\min m(t)|)]} = \frac{|\min m(t)|}{A} = a
\]
The message signal can be recovered from $x_c(t)$ using a technique known as *envelope detection*.

A diode, resistor, and capacitor is all that is needed to construct an envelope detector.

The circuit shown above is actually a combination of a nonlinearity and filter (system with memory).

A detailed analysis of this circuit is more difficult than you might think.

A SPICE circuit simulation is relatively straightforward, but it can be time consuming if $W \ll f_c$.
3.1. LINEAR MODULATION

- The simple envelope detector fails if $A_c[1 + a m_n(t)] < 0$
  - In the circuit shown above, the diode is not ideal and hence there is a turn-on voltage which further limits the maximum value of $a$
- The $RC$ time constant cutoff frequency must lie between both $W$ and $f_c$, hence good operation also requires that $f_c \gg W$
Digital signal processing based envelope detectors are also possible.

Historically the envelope detector has provided a very low-cost means to recover the message signal on AM carrier.

The spectrum of an AM signal is

\[ X_c(f) = \frac{A_c}{2} \left[ \delta(f - f_c) + \delta(f + f_c) \right] \]

pure carrier spectrum

\[ + \frac{a A_c}{2} \left[ M_n(f - f_c) + M_n(f + f_c) \right] \]

DSB spectrum

**AM Power Efficiency**

- Low-cost and easy to implement demodulators is a plus for AM, but what is the downside?

- Adding the bias term to \( m(t) \) means that a fraction of the total transmitted power is dedicated to a pure carrier.

- The total power in \( x_c(t) \) is can be written in terms of the time average operator introduced in Chapter 2

\[ \langle x_c^2(t) \rangle = \langle A_c^2[1 + am_n(t)]^2 \cos^2(2\pi f_c t) \rangle \]

\[ = \frac{A_c^2}{2} \langle [1 + 2am_n(t) + a^2m_n^2(t)][1 + \cos(2\pi(2f_c)t)] \rangle \]

- If \( m(t) \) is slowly varying with respect to \( \cos(2\pi f_c t) \), i.e.,

\[ \langle m(t) \cos \omega_c t \rangle \simeq 0, \]
then

\[ \langle x_c^2(t) \rangle = \frac{A_c^2}{2} \left[ 1 + 2a \langle m_n(t) \rangle + a^2 \langle m_n^2(t) \rangle \right] \]

\[ = \frac{A_c^2}{2} \left[ 1 + a^2 \langle m^2(t) \rangle \right] = \frac{A_c^2}{2} + \frac{a^2 A_c^2}{2} \langle m_n^2(t) \rangle \]

where the last line resulted from the assumption \( \langle m(t) \rangle = 0 \) (the DC or average value of \( m(t) \) is zero)

- **Definition: AM Efficiency**

\[
E_{ff} \triangleq \frac{a^2 \langle m_n^2(t) \rangle}{1 + a^2 \langle m_n^2(t) \rangle} \quad \text{also} \quad \frac{\langle m^2(t) \rangle}{A^2 + \langle m^2(t) \rangle}
\]

---

**Example 3.1: Single Sinusoid AM**

- An AM signal of the form

\[ x_c(t) = A_c [1 + a \cos(2\pi f_m t + \pi/3)] \cos(2\pi f_c t) \]

contains a total power of 1000 W

- The modulation index is 0.8

- Find the power contained in the carrier and the sidebands, also find the efficiency

- The total power is

\[ 1000 = \langle x_c^2(t) \rangle = \frac{A_c^2}{2} + \frac{a^2 A_c^2}{2} \cdot \langle m_n^2(t) \rangle \]
• It should be clear that in this problem \( m_n(t) = \cos(2\pi f_m t) \), so \( \langle m_n^2(t) \rangle = 1/2 \) and

\[
1000 = A_c^2 \left[ \frac{1}{2} + \frac{1}{4} \cdot 0.64 \right] = \frac{33}{50} A_c^2
\]

• Thus we see that

\[
A_c^2 = 1000 \cdot \frac{50}{33} = 1515.15
\]

and

\[
P_{\text{carrier}} = \frac{1}{2} A_c^2 = \frac{1515}{2} = 757.6 \text{ W}
\]

and thus

\[
P_{\text{sidebands}} = 1000 - P_c = 242.4 \text{ W}
\]

• The efficiency is

\[
E_{ff} = \frac{242.4}{1000} = 0.242 \text{ or } 24.2\%
\]

• The magnitude and phase spectra can be plotted by first expanding out \( x_c(t) \)

\[
x_c(t) = A_c \cos(2\pi f_c t) + a A_c \cos(2\pi f_m t + \pi/3) \cos(2\pi f_c t)
\]

\[
= A_c \cos(2\pi f_c t)
\]

\[
+ \frac{a A_c}{2} \cos[2\pi (f_c + f_m) t + \pi/3]
\]

\[
+ \frac{a A_c}{2} \cos[2\pi (f_c - f_m) t - \pi/3]
\]
Example 3.2: Pulse Train with DC Offset

- Find $m_n(t)$ and the efficiency $E$
- From the definition of $m_n(t)$

$$m_n(t) = \frac{m(t)}{|\min m(t)|} = \frac{m(t)}{|-1|} = m(t)$$

- The efficiency is

$$E = \frac{a^2 \langle m_n^2(t) \rangle}{1 + a^2 \langle m_n^2(t) \rangle}$$
To obtain \( \langle m_n^2(t) \rangle \) we form the time average

\[
\langle m_n^2(t) \rangle = \frac{1}{T_m} \left[ \int_0^{T_m/3} (2)^2 \, dt + \int_{T_m/3}^{T_m} (-1)^2 \, dt \right]
\]

\[
= \frac{1}{T_m} \left[ \frac{T_m}{3} \cdot 4 + \frac{2T_m}{3} \cdot 1 \right] = \frac{4}{3} + \frac{2}{3} = \frac{6}{3} = 2
\]

thus

\[
E = \frac{2a^2}{1 + 2a^2}
\]

The best AM efficiency we can achieve with this waveform is when \( a = 1 \)

\[
E_{ff} \bigg|_{a=1} = \frac{2}{3} = 0.67 \text{ or } 67\%
\]

Suppose that the message signal is \(-m(t)\) as given here

Now \( \min m(t) = -2 \) and \( m_n(t) = m(t)/2 \) and

\[
\langle m_n^2(t) \rangle = \frac{1}{3} \cdot (-1)^2 + \frac{2}{3} \cdot (1/2)^2 = \frac{1}{2}
\]

The efficiency in this case is

\[
E_{ff} = \frac{(1/2)a^2}{1 + (1/2)a^2} = \frac{a^2}{2 + a^2}
\]

Now when \( a = 1 \) we have \( E_{ff} = 1/3 \) or just 33.3%

Note that for 50% duty cycle squarewave the efficiency maximum is just 50%
Example 3.3: Multiple Sinusoids

- Suppose that $m(t)$ is a sum of multiple sinusoids (multi-tone AM)

$$m(t) = \sum_{k=1}^{M} A_k \cos(2\pi f_k t + \phi_k)$$

where $M$ is the number of sinusoids, $f_k$ values might be constrained over some band of frequencies $W$, e.g., $f_k \leq W$, and the phase values $\phi_k$ can be any value on $[0, 2\pi]$

- To find $m_n(t)$ we need to find $\min m(t)$

- A lower bound on $\min m(t)$ is $-\sum_{k=1}^{M} A_k$; why?

- The worst case value may not occur in practice depending upon the phase and frequency values, so we may have to resort to a numerical search or a plot of the waveform

- Suppose that $M = 3$ with $f_k = \{65, 100, 35\}$ Hz, $A_k = \{2, 3.5, 4.2\}$, and $\phi_k = \{0, \pi/3, -\pi/4\}$ rad.

\[
\text{>> [m,t] = M_sinusoids(1000,[65 100 35],[2 3.5 4.2],...}
\text{[0 pi/3 -pi/4], 20000);>> plot(t,m)}
\]

\[
\text{>> min(m)}
\]

\[
\text{ans = -7.2462e+00}
\]

\[
\text{>> -sum([2 3.5 4.2]) \% worst case minimum value}
\]

\[
\text{ans = -9.7000e+00}
\]

\[
\text{>> subplot(311)}
\]

\[
\text{>> plot(t,(1 + 0.25*m/abs(min(m))).*cos(2*pi*1000*t))}
\]

\[
\text{>> hold}
\]
Current plot held
>> plot(t,1 + 0.25*m/abs(min(m)),'r')
>> subplot(312)
>> plot(t,(1 + 0.5*m/abs(min(m))).*cos(2*pi*1000*t))
>> hold
Current plot held
>> plot(t,1 + 0.5*m/abs(min(m)),'r')
>> subplot(313)
>> plot(t,(1 + 1.0*m/abs(min(m))).*cos(2*pi*1000*t))
>> hold
Current plot held
>> plot(t,1 + 1.0*m/abs(min(m)),'r')

Finding $\min m(t)$ graphically

- The normalization factor is approximately given by 7.246, that is
  \[
  m_n(t) = \frac{m(t)}{7.246}
  \]
- Shown below are plots of $x_c(t)$ for $a = 0.25, 0.5$ and 1 using $f_c = 1000$ Hz
3.1. LINEAR MODULATION

- To obtain the efficiency of multi-tone AM we first calculate $\langle m_n^2(t) \rangle$ assuming unique frequencies

$$\langle m_n^2(t) \rangle = \sum_{k=1}^{M} \frac{A_k^2}{2 \left| \min m(t) \right|^2}$$

$$= \frac{2^2 + 3.5^2 + 4.2^2}{2 \times 7.246^2} = 0.3227$$

- The maximum efficiency is just

$$E_{ff} \bigg|_{a=1} = \frac{0.3227}{1 + 0.3227} = 0.244 \text{ or } 24.4\%$$
A remaining interest is the spectrum of $x_c(t)$

$$X_c(f) = \frac{A_c}{2} \left[ \delta(f - f_c) + \delta(f + f_c) \right]$$

$$+ \frac{a A_c}{4} \sum_{k=1}^{M} A_k \left[ e^{j\phi_k} \delta(f - (f_c + f_k)) \right.$$

$$+ e^{-j\phi_k} \delta(f + (f_c + f_k)) \bigg] \text{ (USB terms)}$$

$$+ \frac{a A_c}{4} \sum_{k=1}^{M} A_k \left[ e^{j\phi_k} \delta(f - (f_c - f_k)) \right.$$  

$$+ e^{-j\phi_k} \delta(f + (f_c - f_k)) \bigg] \text{ (LSB terms)}$$

Amplitude spectra

Carrier with $A_c = 1$

Symmetrical Sidebands for $a = 0.5$
3.1.3 Single-Sideband Modulation

- In the study of DSB it was observed that the USB and LSB spectra are related, that is the magnitude spectra about $f_c$ has even symmetry and phase spectra about $f_c$ has odd symmetry.

- The information is redundant, meaning that $m(t)$ can be reconstructed from one or the other sidebands.

- Transmitting just the USB or LSB results in single-sideband (SSB).

- For $m(t)$ having lowpass bandwidth of $W$ the bandwidth required for DSB, centered on $f_c$ is $2W$.

- Since SSB operates by transmitting just one sideband, the transmission bandwidth is reduced to just $W$.

DSB to two forms of SSB: USSB and LSSB:

- The filtering required to obtain an SSB signal is best explained with the aid of the Hilbert transform, so we divert from text.
Hilbert Transform

- The Hilbert transform is nothing more than a filter that shifts the phase of all frequency components by $-\pi/2$, i.e.,

$$H(f) = -j\text{sgn}(f)$$

where

$$\text{sgn}(f) = \begin{cases} 
1, & f > 0 \\
0, & f = 0 \\
-1, & f < 0 
\end{cases}$$

- The Hilbert transform of signal $x(t)$ can be written in terms of the Fourier transform and inverse Fourier transform

$$\hat{x}(t) = \mathcal{F}^{-1}[ -j\text{sgn}(f)X(f) ] = h(t) * x(t)$$

where $h(t) = \mathcal{F}^{-1}\{H(f)\}$

- We can find the impulse response $h(t)$ using the duality theorem and the differentiation theorem

$$\frac{d}{df}H(f) \leftrightarrow (-j2\pi t)h(-t)$$

where here $H(f) = -j\text{sgn}(f)$, so

$$\frac{d}{df}H(f) = -2j\delta(f)$$
3.1. LINEAR MODULATION

- Clearly,

\[ \mathcal{F}^{-1}\{ -2j \delta(f) \} = -2j \]

so

\[ h(t) = \frac{-2j}{-j 2\pi t} = \frac{1}{\pi t} \]

and

\[ \frac{1}{\pi t} \xrightarrow{\mathcal{F}} -j \text{sgn}(f) \]

- In the time domain the Hilbert transform is the convolution integral

\[ \hat{x}(t) = \int_{-\infty}^{\infty} \frac{x(\lambda)}{\pi(t-\lambda)} \, d\lambda = \int_{-\infty}^{\infty} \frac{x(t-\lambda)}{\pi \lambda} \, d\lambda \]

- Note that since the Hilbert transform of \( x(t) \) is a \(-\pi/2\) phase shift, the Hilbert transform of \( \hat{x}(t) \) is

\[ \hat{\hat{x}}(t) = -x(t) \]

why? observe that \((-j \text{sgn}(f))^2 = -1\)

Example 3.4: \( x(t) = \cos \omega_0 t \)

- By definition

\[ \hat{X}(f) = -j \text{sgn}(f) \cdot \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] \]

\[ = -j \frac{1}{2} \delta(f - f_0) + j \frac{1}{2} \delta(f + f_0) \]
so from $e^{j\omega_0 t} \xrightarrow{\mathcal{F}} \delta(f - f_0)$

$$\hat{x}(t) = -j \frac{1}{2} e^{j\omega_0 t} + j \frac{1}{2} e^{-j\omega_0 t}$$

$$= \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \sin \omega_0 t$$

or

$$\overline{\cos \omega_0 t} = \sin \omega_0 t$$

- It also follows that

$$\overline{\sin \omega_0 t} = \overline{\cos \omega_0 t} = -\cos \omega_0 t$$

since $\hat{x}(t) = -x(t)$

---

**Hilbert Transform Properties**

1. The energy (power) in $x(t)$ and $\hat{x}(t)$ are equal

   The proof follows from the fact that $|Y(f)|^2 = |H(f)|^2 |X(f)|^2$

   and $|j \text{sgn}(f)|^2 = 1$

2. $x(t)$ and $\hat{x}(t)$ are orthogonal, that is

   $$\int_{-\infty}^{\infty} x(t)\hat{x}(t) \, dt = 0 \text{ (energy signal)}$$

   $$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\hat{x}(t) \, dt = 0 \text{ (power signal)}$$
The proof follows for the case of energy signals by generalizing Parseval’s theorem

\[
\int_{-\infty}^{\infty} x(t)\hat{x}(t) \, dt = \int_{-\infty}^{\infty} X(f)\hat{X}^*(f) \, df
\]

\[
= \int_{-\infty}^{\infty} \left( j \text{sgn}(f) \right) |X(f)|^2 \, df = 0
\]

3. Given signals \( m(t) \) and \( c(t) \) such that the corresponding spectra are

\[
M(f) = 0 \text{ for } |f| > W \text{ (a lowpass signal)}
\]

\[
C(f) = 0 \text{ for } |f| < W \text{ (c(t) a highpass signal)}
\]

then

\[
\overline{m(t)c(t)} = m(t)\hat{c}(t)
\]

**Example 3.5:** \( c(t) = \cos \omega_0 t \)

- Suppose that \( M(f) = 0 \) for \( |f| > W \) and \( f_0 > W \) then

\[
\overline{m(t) \cos \omega_0 t} = m(t)\cos \omega_0 t
\]

\[= m(t) \sin \omega_0 t
\]

**Analytic Signals**

- Define analytic signal \( z(t) \) as

\[
z(t) = x(t) + j\hat{x}(t)
\]

where \( x(t) \) is a real signal
• The *envelope* of $z(t)$ is $|z(t)|$ and is related to the envelope discussed with DSB and AM signals.

• The spectrum of an analytic signal has single-sideband characteristics.

• In particular for $z_p(t) = x(t) + j\hat{x}(t)$

\[
Z_p(f) = X(f) + j\{-j\text{sgn}(f)X(f)\} \\
= X(f)[1 + \text{sgn}(f)] \\
= \begin{cases} 
2X(f), & f > 0 \\
0, & f < 0 
\end{cases}
\]

**Note:** Only positive frequencies present.

• Similarly for $z_n(t) = x(t) - j\hat{x}(t)$

\[
Z_n(f) = X(f)[1 - \text{sgn}(f)] \\
= \begin{cases} 
0, & f > 0 \\
2X(f), & f < 0 
\end{cases}
\]
3.1. LINEAR MODULATION

The spectra of analytic signals can suppress positive or negative frequencies.

Return to SSB Development

Basic SSB signal generation

- In simple terms, we create an SSB signal from a DSB signal using a sideband filter

- The mathematical representation of LSSB and USSB signals makes use of Hilbert transform concepts and analytic signals
From the frequency domain expression for the LSSB, we can ultimately obtain an expression for the LSSB signal, $x_{\text{clSSB}}(t)$, in the time domain.

Start with $X_{\text{DSB}}(f)$ and the filter $H_L(f)$:

$$X_{\text{clSSB}}(f) = \frac{1}{2} A_c [M(f + f_c) + M(f - f_c)]$$

$$\cdot \frac{1}{2} \left[ \text{sgn}(f + f_c) - \text{sgn}(f - f_c) \right]$$
The inverse Fourier transform of the second term is DSB, i.e.,
\[
\frac{1}{2}A_c m(t) \cos \omega_c t \leftrightarrow \mathcal{F} \left\{ \frac{1}{4}A_c [M(f + f_c) + M(f - f_c)] \right\}
\]

The first term can be inverse transformed using
\[
\hat{m}(t) \leftrightarrow -j \text{sgn}(f) \cdot M(f)
\]
so
\[
\mathcal{F}^{-1}\{M(f + f_c)\text{sgn}(f + f_c)\} = j\hat{m}(t)e^{-j\omega_c t}
\]
since \(m(t)e^{\pm j\omega_c t} \leftrightarrow \mathcal{F} M(f \pm f_c)\)

Thus
\[
\frac{1}{4}A_c \mathcal{F}^{-1}\{M(f + f_c)\text{sgn}(f + f_c) - M(f - f_c)\text{sgn}(f - f_c)\}
\]
\[
= \frac{1}{4}A_c [j\hat{m}(t)e^{-j\omega_c t} - j\hat{m}(t)e^{j\omega_c t}] = \frac{1}{2}\hat{m}(t) \sin \omega_c t
\]
Finally, 

\[ x_{cLSSB}(t) = \frac{1}{2} A_c m(t) \cos \omega_c t + \frac{1}{2} A_c \hat{m}(t) \sin \omega_c t \]

Similarly for USSB it can be shown that 

\[ x_{cUSSB}(t) = \frac{1}{2} A_c m(t) \cos \omega_c t - \frac{1}{2} A_c \hat{m}(t) \sin \omega_c t \]

The direct implementation of SSB is very difficult due to the requirements of the filter.

By moving the phase shift frequency from \( f_c \) down to DC (0 Hz) the implementation is much more reasonable (this applies to a DSP implementation as well).

The phase shift is not perfect at low frequencies, so the modulation must not contain critical information at these frequencies.

Phase shift modulator for SSB
Demodulation

- The coherent demodulator first discussed for DSB, also works for SSB

\[ d(t) = \frac{1}{2} A_c [m(t) \cos \omega_c t \pm \hat{m}(t) \sin \omega_c t] 4 \cos(\omega_c t + \theta(t)) \]

\[ = A_c m(t) \cos \theta(t) + A_c m(t) \cos[2\omega_c t + \theta(t)] \]

\[ + A_c \hat{m}(t) \sin \theta(t) \pm A_c \hat{m}(t) \sin[2\omega_c t + \theta(t)] \]

so

\[ y_D(t) = m(t) \cos \theta(t) \mp \hat{m}(t) \sin \theta(t) \]

\[ \theta(t) \text{ small} \quad \simeq \quad m(t) \mp \hat{m}(t) \theta(t) \]

- The \( \hat{m}(t) \sin \theta(t) \) term represents crosstalk

- Another approach to demodulation is to use carrier reinsertion and envelope detection

\[ x_r(t) \overset{\text{+}}{\longrightarrow} e(t) \overset{\text{Envelope Detector}}{\longrightarrow} y_D(t) \]

\[ K \cos \omega_c t \]
\[ e(t) = x_r(t) + K \cos \omega_c t \]

\[ = \left[ \frac{1}{2} A_c m(t) + K \right] \cos \omega_c t \pm \frac{1}{2} A_c \hat{m}(t) \sin \omega_c t \]

- To proceed with the analysis we must find the envelope of \( e(t) \), which will be the final output \( y_D(t) \)

- Finding the envelope is a more general problem which will be useful in future problem solving, so first consider the envelope of

\[ x(t) = \left\{ \begin{array}{l}
\frac{1}{2} \cos \omega_c t - b(t) \sin \omega_c t \\
\sin \omega_c t + a(t) \cos \omega_c t
\end{array} \right. \]

\[ = \Re \left\{ \left[ a(t) + j b(t) \right] e^{j \omega_c t} \right\} \]

\[ = \tilde{R}(t) \]

- In a \textit{phasor diagram} \( x(t) \) consists of an \textit{inphase} or direct component and a \textit{quadrature} component

Note: \( \tilde{R}(t) = R(t) e^{j \theta(t)} = a(t) + j b(t) \)
where the resultant $R(t)$ is such that

$$a(t) = R(t) \cos \theta(t)$$
$$b(t) = R(t) \sin \theta(t)$$

which implies that

$$x(t) = R(t) \left[ \cos \theta(t) \cos \omega_c t - \sin \theta(t) \sin \omega_c t \right]$$

$$= R(t) \cos \left[ \omega_c t + \theta(t) \right]$$

where $\theta(t) = \tan^{-1}[b(t)/a(t)]$

- The signal envelope is thus given by

$$R(t) = \sqrt{a^2(t) + b^2(t)}$$

- The output of an envelope detector will be $R(t)$ if $a(t)$ and $b(t)$ are slowly varying with respect to $\cos \omega_c t$

- In the SSB demodulator

$$y_D(t) = \sqrt{\left[ \frac{1}{2} A_c m(t) + K \right]^2 + \left[ \frac{1}{2} A_c \hat{m}(t) \right]^2}$$

- If we choose $K$ such that $(A_c m(t)/2 + K)^2 \gg (A_c \hat{m}(t)/2)^2$, then

$$y_D(t) \approx \frac{1}{2} A_c m(t) + K$$

- **Note:**
  - The above analysis assumed a phase coherent reference
  - In speech systems the frequency and phase can be adjusted to obtain intelligibility, but not so in data systems
The approximation relies on the binomial expansion

\[(1 + x)^{1/2} \simeq 1 + \frac{1}{2}x \text{ for } |x| \ll 1\]

**Example 3.6: Noncoherent Carrier Reinsertion**

- Let \(m(t) = \cos \omega_m t, \omega_m \ll \omega_c\) and the reinserted carrier be \(K \cos[(\omega_c + \Delta \omega)t]\)

- Following carrier reinsertion we have

\[
e(t) = \frac{1}{2} A_c \cos \omega_m t \cos \omega_c t
+ \frac{1}{2} A_c \sin \omega_m t \sin \omega_c t + K \cos \left[(\omega_c + \Delta \omega)t\right]
= \frac{1}{2} A_c \cos \left[(\omega_c \pm \omega_m)t\right] + K \cos \left[(\omega_c + \Delta \omega)t\right]
\]

- We can write \(e(t)\) as the real part of a complex envelope times a carrier at either \(\omega_c\) or \(\omega_c + \Delta \omega\)

- In this case, since \(K\) will be large compared to \(A_c/2\), we write

\[
e(t) = \frac{1}{2} A_c \text{Re}\left\{e^{\pm j\omega_m t} e^{j\omega_c t}\right\}
+ K \text{Re}\left\{1 \cdot e^{j(\omega_c + \Delta \omega)t}\right\}
= \text{Re}\left\{\left(\frac{1}{2} A_c e^{j(\pm \omega_m - \Delta \omega)t} + K\right)e^{j(\omega_c + \Delta \omega)t}\right\}
\]

complex envelope \(\tilde{R}(t)\)
• Finally expanding the complex envelope into the real and imaginary parts we can find the real envelope $R(t)$

$$y_D(t) = \left[ \left\{ \frac{1}{2} A_c \cos[\pm \omega_m + \Delta \omega] t + K \right\}^2 + \left\{ \frac{1}{2} A_c \sin[\pm \omega_m + \Delta \omega] t \right\}^2 \right]^{1/2}$$

$$\approx \frac{1}{2} A_c \cos[(\omega_m \mp \Delta \omega)t] + K$$

where the last line follows for $K \gg A_c$

• Note that the frequency error $\Delta \omega$ causes the recovered message signal to shift up or down in frequency by $\Delta \omega$, but not both at the same time as in DSB, thus the recovered speech signal is more intelligible

3.1.4 Vestigial-Sideband Modulation

• Vestigial sideband (VSB) is derived by filtering DSB such that one sideband is passed completely while only a vestige remains of the other

• Why VSB?

1. Simplifies the filter design
2. Improves the low-frequency response and allows DC to pass undistorted
3. Has bandwidth efficiency advantages over DSB or AM, similar to that of SSB
- A primary application of VSB is the video portion of analog television (note HDTV replaces this in the US with 8VSB\(^1\))

- The generation of VSB starts with DSB followed by a filter that has a \(2\beta\) transition band, e.g.,

\[
|H(f)| = \begin{cases} 
0, & f < F_c - \beta \\
\frac{f-(f_c-\beta)}{2\beta}, & f_c - \beta \leq f \leq f_c + \beta \\
1, & f > f_c + \beta
\end{cases}
\]

- VSB can be demodulated using a coherent demod or using carrier reinsertion and envelope detection

\[\text{Transmitted Two-Tone Spectrum (only single-sided shown)}\]

\[\text{Two-tone VSB signal}\]

\(^1\)http://www.tek.com/document/primer/fundamentals-8vsb
• Suppose the message signal consists of two tones

\[ m(t) = A \cos \omega_1 t + B \cos \omega_2 t \]

• Following the DSB modulation and VSB shaping,

\[ x_c(t) = \frac{1}{2} A \epsilon \cos(\omega_c - \omega_1) t \]
\[ + \frac{1}{2} A (1 - \epsilon) \cos(\omega_c + \omega_1) t + \frac{1}{2} B \cos(\omega_c + \omega_2) t \]

• A coherent demod multiplies the received signal by \( 4 \cos \omega_c t \) to produce

\[ e(t) = A \epsilon \cos \omega_1 t + A (1 - \epsilon) \cos \omega_1 t + B \cos \omega_2 t \]
\[ = A \cos \omega_1 t + B \cos \omega_2 t \]

which is the original message signal

• The symmetry of the VSB shaping filter has made this possible

• In the case of broadcast TV the carrier in included at the transmitter to insure phase coherency and easy demodulation at the TV receiver (VSB + Carrier)

  – Very large video carrier power was required for typical TV station, i.e., greater than 100,000 W
  
  – To make matters easier still, the precise VSB filtering is not performed at the transmitter due to the high power requirements, instead the TV receiver did this
  
  – Further study is needed on today’s 8VSB
3.1.5 Frequency Translation and Mixing

- Used to translate baseband or bandpass signals to some new center frequency

Local oscillator of the form
\[ e(t) = m(t) \cos(\omega_1 t) \]

Frequency translation system

- Assuming the input signal is DSB of bandwidth \( 2W \) the mixer (multiplier) output is

\[
e(t) = m(t) \cos(\omega_1 t) \cdot 2 \cos(\omega_1 \pm \omega_2) t = m(t) \cos(\omega_2 t) + m(t) \cos[(2\omega_1 \pm \omega_2) t]
\]
3.1. LINEAR MODULATION

- The bandpass filter bandwidth needs to be at least $2W$ Hz wide.
- Note that if an input of the form $k(t) \cos((\omega_1 \pm 2\omega_2)t)$ is present it will be converted to $\omega_2$ also, i.e.,

$$e(t) = k(t) \cos(\omega_2 t) + k(t) \cos((2\omega_1 \pm 3\omega_2)t),$$

and the bandpass filter output is $k(t) \cos(\omega_2 t)$.

- The frequencies $\omega_1 \pm 2\omega_2$ are the image frequencies of $\omega_1$ with respect to $\omega_{LO} = \omega_1 \pm \omega_2$.

---

**Example 3.7: AM Broadcast Superheterodyne Receiver**

![Diagram of AM Broadcast Superheterodyne Receiver]

- $2f_{IF} > B_{RF} > B_T$
- $B_{IF} \approx B_T$
- $B_{AF} = W$
- For AM $B_T = 2W$

**AM Broadcast Specs:** $f_c = 540$ to $1600$ kHz on 10 kHz spacings
- Carrier stability ±$20$ Hz
- Modulated audio flat 100 Hz to 5 kHz
- Typical $f_{IF} = 455$ kHz

Classical AM superheterodyne receiver

- We have two choices for the local oscillator, *high-side* or *low-side* tuning.
- Low-side: $540 - 455 \leq f_{LO} \leq 1600 - 455$ or $85 \leq f_{LO} \leq 1145$, all frequencies in kHz

- High-side: $540 + 455 \leq f_{LO} \leq 1600 + 455$ or $995 \leq f_{LO} \leq 2055$, all frequencies in kHz

- The high-side option is advantageous since the tunable oscillator or frequency synthesizer has the smallest frequency ratio $f_{LO,max}/f_{LO,min} = 2055/995 = 2.15$

- Suppose the desired station is at 560 kHz, then with high-side tuning we have $f_{LO} = 560 + 455 = 1015$ kHz

- The image frequency is at $f_{image} = f_c + 2f_{IF} = 560 + 2 \times 455 = 1470$ kHz (note this is another AM radio station center frequency)
Example 3.8: A Double-Conversion Receiver

\[ f_c = 162.475 \text{ MHz} \]  
(WX #4)

\[ f_{\text{LO}_1} = 173.175 \text{ MHz} \]  
\[ f_{\text{LO}_2} = 11.155 \text{ MHz} \]

Double-conversion superheterodyne receiver (Lab 4)

- Consider a frequency modulation (FM) receiver that uses double-conversion to receive a signal on carrier frequency 162.475 MHz (weather channel #4 here in Colorado Springs)
  - Frequency modulation will be discussed in the next section

- The dual-conversion allows good image rejection by using a 10.7 MHz first IF and then can provide good selectivity by using a second IF at 455 kHz; why?
  - The ratio of bandwidth to center frequency can only be so small in a low loss RF filter
  - The second IF filter can thus have a much narrower bandwidth by virtue of the center frequency being much lower

- A higher first IF center frequency moves the image signal further away from the desired signal
– For high-side tuning we have $f_{\text{image}} = f_c + 2f_{\text{IF}} = f_c + 21.4$ MHz

- Double-conversion receivers are more complex to implement

---

**Mixers**

- The multiplier that is used to implement frequency translation is often referred to as a mixer

- In the world of RF circuit design the term mixer is more appropriate, as an ideal multiplier is rarely available

- Instead active and passive circuits that approximate signal multiplication are utilized

- The notion of mixing comes about from passing the sum of two signals through a nonlinearity, e.g.,

$$y(t) = [a_1 x_1(t) + a_2 x_2(t)]^2 + \text{other terms}$$

$$= a_1^2 x_1^2(t) + 2a_1 a_2 x_1(t)x_2(t) + a_2^2 x_2^2(t)$$

- In this mixing application we are most interested in the center term

$$y_{\text{desired}}(t) = 2a_1 a_2 [x_1(t) \cdot x_2(t)]$$

- Clearly this mixer produces unwanted terms (first and third), and in general many other terms, since the nonlinearity will have more than just a square-law input/output characteristic
3.1. LINEAR MODULATION

- A diode or active device can be used to form mixing products as described above, consider the dual-gate MEtal Semiconductor FET (MESFET) mixer shown below

![Mixer concept](image)

- The double-balanced mixer (DBM), which can be constructed using a diode ring, provides better isolation between the RF, LO, and IF ports

- When properly balanced the DBM also allows even harmonics to be suppressed in the mixing operation
• A basic transformer coupled DBM, employing a diode ring, is shown below, followed by an active version

• The DBM is suitable for use as a *phase detector* in phase-locked loop applications

825 MHz to 915 MHz SiGe High-Linearity Active DBM
Example 3.9: Single Diode Mixer

- **AM Signal at 560 kHz, 10 kHz Tone Message**
- **Signal Sum**
- **Mixing Diode**
- **VI Probe for PSD**

**ADS single diode mixer simulation:** 560 kHz → 455 kHz
3.2 Interference

Interference is a fact of life in communication systems. A thorough understanding of interference requires a background in random signals analysis (Chapter 6 of the text), but some basic concepts can be obtained by considering a single interference at $f_c + f_i$ that lies close to the carrier $f_c$.

3.2.1 Interference in Linear Modulation

![Single-Sided Spectrum](image)

AM carrier with single tone interference

- If a single tone carrier falls within the IF passband of the receiver what problems does it cause?

- Coherent Demodulator

$$x_r(t) = [A_c \cos \omega_c t + A_m \cos \omega_m t \cos \omega_c t] + A_i \cos (\omega_c + \omega_i)t$$

  - We multiply $x_r(t)$ by $2 \cos \omega_c t$ and lowpass filter

$$y_D(t) = A_m \cos \omega_m t + A_i \cos \omega_i t$$

- Envelope Detection: Here we need to find the received envelope relative to the strongest signal present.
- **Case** $A_c \gg A_i$

- We will expand $x_r(t)$ in complex envelope form by first noting that

$$A_i \cos(\omega_c + \omega_i) t = A_i \cos \omega_i t \cos \omega_c t - A_i \sin \omega_i t \sin \omega_c t$$

now,

$$x_r(t) = \text{Re}\{(A_c + A_m \cos \omega_m t + A_i \cos \omega_i t$$

$$- jA_i \sin \omega_i t) e^{j\omega_c t}\}$$

$$= \text{Re}\{\tilde{R}(t)e^{j\omega_c t}\}$$

so

$$R(t) = |\tilde{R}(t)|$$

$$= \left[(A_c + A_m \cos \omega_m t + A_i \cos \omega_i t)^2$$

$$+ (A_i \sin \omega_i t)^2\right]^{1/2}$$

$$\approx A_c + A_m \cos \omega_m t + A_i \cos \omega_i t$$

assuming that $A_c \gg A_i$

- Finally,

$$y_D(t) \approx A_m \cos \omega_m t + \underbrace{A_i \cos \omega_i t}_{\text{interference}}$$

- **Case** $A_i \gg A_c$

- Now the interfering term looks like the carrier and the remaining terms look like sidebands, LSSB sidebands relative to $f_c + f_i$ to be specific
– From SSB envelope detector analysis we expect

\[ y_D(t) \simeq \frac{1}{2} A_m \cos(\omega_i + \omega_m)t + A_c \cos \omega_i t \]

\[ + \frac{1}{2} A_m \cos(\omega_i - \omega_m)t \]

and we conclude that the message signal is lost!
3.3 Sampling Theory

- We now return to text Chapter 2, Section 8, for an introduction/review of sampling theory.

- Consider the representation of continuous-time signal \( x(t) \) by the sampled waveform

\[
x_\delta(t) = x(t) \left[ \sum_{n=-\infty}^\infty \delta(t - nT_s) \right] = \sum_{n=-\infty}^\infty x(nT_s) \delta(t - nT_s)
\]

- How is \( T_s \) selected so that \( x(t) \) can be recovered from \( x_\delta(t) \)?

- **Uniform Sampling Theorem for Lowpass Signals**

Given

\[
\mathcal{F}\{x(t)\} = X(f) = 0, \quad \text{for } f > W
\]

then choose

\[
T_s < \frac{1}{2W} \quad \text{or} \quad f_s > 2W \quad (f_s = 1/T_s)
\]

to reconstruct \( x(t) \) from \( x_\delta(t) \) and pass \( x_\delta(t) \) through an ideal LPF with cutoff frequency \( W < B < f_s - W \)

- \( 2W = \text{Nyquist frequency} \)
- \( f_s/2 = \text{folding frequency} \)
proof:

\[ X_\delta(f) = X(f) \ast \left[ f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right] \]

but \( X(f) \ast \delta(f - nf_s) = X(f - nf_s) \), so

\[ X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \]

Spectra before and after sampling at rate \( f_s \):

- As long as \( f_s - W > W \) or \( f_s > 2W \) there is no **aliasing** (spectral overlap)
• To recover \( x(t) \) from \( x_\delta(t) \) all we need to do is lowpass filter the sampled signal with an ideal lowpass filter having cutoff frequency \( W < f_{\text{cutoff}} < f_s - W \)

• In simple terms we set the lowpass bandwidth to the folding frequency, \( f_s/2 \)

• Suppose the reconstruction filter is of the form

\[
H(f) = H_0 \prod \left( \frac{f}{2B} \right) e^{-j2\pi f t_0}
\]

we then choose \( W < B < f_s - W \)

• For input \( X_\delta(f) \), the output spectrum is

\[
Y(f) = f_s H_0 X(f)e^{-j2\pi f t_0}
\]

and in the time domain

\[
y(t) = f_s H_0 x(t - t_0)
\]

• If the reconstruction filter is not ideal we then have to design the filter in such a way that minimal desired signal energy is removed, yet also minimizing the contributions from the spectral translates either side of the \( n = 0 \) translate

• The reconstruction operation can also be viewed as interpolating signal values between the available sample values

• Suppose that the reconstruction filter has impulse response \( h(t) \),
then

\[ y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h(t - nT_s) \]

\[ = 2BH_0 \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}[2B(t - t_0 - nT_s)] \]

where in the last lines we invoked the ideal filter described earlier

- **Uniform Sampling Theorem for Bandpass Signals**

  If \( x(t) \) has a single-sided bandwidth of \( W \) Hz and

  \[ \mathcal{F}\{x(t)\} = 0 \text{ for } f > f_u \]

  then we may choose

  \[ f_s = \frac{2f_u}{m} \]

  where

  \[ m = \left\lfloor \frac{f_u}{W} \right\rfloor, \]

  which is the greatest integer less than or equal to \( f_u/W \)

---

**Example 3.10:** Bandpass signal sampling

![Input signal spectrum](image)

Input signal spectrum
• In the above signal spectrum we see that

\[ W = 2, \quad f_u = 4 \quad f_u/W = 2 \Rightarrow m = 2 \]

so

\[ f_s = \frac{2(4)}{2} = 4 \]

will work

• The sampled signal spectrum is

\[ X_\delta(f) = 4 \sum_{n=-\infty}^{\infty} X(f - nf_s) \]
3.4 Analog Pulse Modulation

- The message signal $m(t)$ is sampled at rate $f_s = 1/T_s$
- A characteristic of the transmitted pulse is made to vary in a one-to-one correspondence with samples of the message signal
- A digital variation is to allow the pulse attribute to take on values from a finite set of allowable values

3.4.1 Pulse-Amplitude Modulation (PAM)

- PAM produces a sequence of flat-topped pulses whose amplitude varies in proportion to samples of the message signal
- Start with a message signal, $m(t)$, that has been uniformly sampled

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s)$$

- The PAM signal is

$$m_c(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\Pi\left(\frac{t - (nT_s + \tau/2)}{\tau}\right)$$
3.4. ANALOG PULSE MODULATION

- It is possible to create $m_c(t)$ directly from $m_\delta(t)$ using a zero-order hold filter, which has impulse response

$$h(t) = \prod \left( \frac{t - \tau/2}{\tau} \right)$$

and frequency response

$$H(f) = \tau \text{sinc}(f\tau)e^{-j\pi f\tau}$$

- How does $h(t)$ change the recovery operation from the case of ideal sampling?
  - If $\tau \ll T_s$ we can get by with just a lowpass reconstruction filter having cutoff frequency at $f_s/2 = 2/T_s$
  - In general, there may be a need for equalization if $\tau$ is on the order of $T_s/4$ to $T_s/2$
3.4.2 Pulse-Width Modulation (PWM)

- A PWM waveform consists of pulses with width proportional to the sampled analog waveform.
- For bipolar $m(t)$ signals we may choose a pulse width of $T_s/2$ to correspond to $m(t) = 0$.
- The biggest application for PWM is in motor control.
- It is also used in class $D$ audio power amplifiers.
- A lowpass filter applied to a PWM waveform recovers the modulation $m(t)$.

![Example PWM signal](image)

3.4.3 Pulse-Position Modulation

- With PPM the displacement in time of each pulse, with respect to a reference time, is proportional to the sampled analog waveform.
- The time axis may be slotted into a discrete number of pulse positions, then $m(t)$ would be quantized.
- Digital modulation that employs $M$ slots, using nonoverlapping pulses, is a form of $M$-ary orthogonal communications.
3.5. DELTA MODULATION AND PCM

- PPM of this type is finding application in *ultra-wideband communications*

![Example PPM signal]

Example PPM signal

3.5 Delta Modulation and PCM

- This section considers two pure digital pulse modulation schemes

- Pure digital means that the output of the modulator is a binary waveform taking on only discrete values

3.5.1 Delta Modulation (DM)

- The message signal $m(t)$ is encoded into a binary sequence which corresponds to changes in $m(t)$ relative to reference waveform $m_s(t)$

- DM gets its name from the fact that only the difference from sample-to-sample is encoded

- The sampling rate in combination with the step size are the two primary controlling modulator design parameters
\[ \delta_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \]

The maximum slope that can be followed is \( \delta_0 / T_s \)

Delta modulator with step size parameter \( \delta_0 \)

Start-up transient

Slope overload

\[ \delta_0 = 0.15 \]

\[ \sum_{n=-\infty}^{\infty} \delta_0 \Delta(nT_s) u(t - nT_s) \]

\[ \int \Delta(t) \, dt \]

Control the step size

\[ m(t) \]

\[ m_s(t) = x_c(t) \Delta(t) \]

\[ d(t) \]

\[ m(t) \text{ (blue)} \]

\[ m_s(t) \text{ (red)} \]

\[ \delta_0 = 0.15 \]

\[ \text{Slope overload} \]

\[ \text{Start-up transient} \]

Delta modulator waveforms
A MATLAB DM simulation function is given below

```matlab
function [t_o,x,ms] = DeltaMod(m,fs,delta_0,L)

n = 0:(L*length(m))-1;
t_o = n/(L*fs);
ms = zeros(size(m));
x = zeros(size(m));
ms_old = 0; % zero initial condition
for k=1:length(m)
    x(k) = sign(m(k) - ms_old);
    ms(k) = ms_old + x(k)*delta_0;
    ms_old = ms(k);
end
x = [x; zeros(L-1,length(m))];
x = reshape(x,1,L*length(m));
```

The message \( m(t) \) can be recovered from \( x_c(t) \) by integrating and then lowpass filtering to remove the stair step edges (low-pass filtering directly is a simplification)

Slope overload can be dealt with through an adaptive scheme

- If \( m(t) \) is nearly constant keep the step size \( \delta_0 \) small
- If \( m(t) \) has large variations, a larger step size is needed

With adaptive DM the step size is controlled via a variable gain amplifier, where the gain is controlled by square-law detecting the output of a lowpass filter acting on \( x_c(t) \)
\[ \delta_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \]

Means to obtain a variable step size DM

### 3.5.2 Pulse-Code Modulation (PCM)

- Each sample of \( m(t) \) is mapped to a binary word by

1. Sampling
2. Quantizing
3. Encoding
3.5. DELTA MODULATION AND PCM

<table>
<thead>
<tr>
<th>Quant. Level</th>
<th>Encoded Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>111</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
</tr>
<tr>
<td>3</td>
<td>011</td>
</tr>
<tr>
<td>2</td>
<td>010</td>
</tr>
<tr>
<td>1</td>
<td>001</td>
</tr>
<tr>
<td>0</td>
<td>000</td>
</tr>
</tbody>
</table>

Quantizer Bits: $n = 3$, $q = 2^n = 8$

Encoded Serial PCM Data: 001 100 110 111 110 100 010 010 ...

3-Bit PCM encoding

- Assume that $m(t)$ has bandwidth $W$ Hz, then
  - Choose $f_s > 2W$
  - Choose $n$ bits per sample ($q = 2^n$ quantization levels)
  - $\Rightarrow 2nW$ binary digits per second must be transmitted

- Each pulse has width no more than
  $$(\Delta \tau)_{\text{max}} = \frac{1}{2nW},$$

so using the fact that the lowpass bandwidth of a single pulse is about $1/(2\Delta \tau)$ Hz, we have that the lowpass transmission bandwidth for PCM is approximately

$$B \simeq kWn,$$

where $k$ is a proportionality constant

- When located on a carrier the required bandwidth is doubled
Binary phase-shift keying (BPSK), mentioned earlier, is a popular scheme for transmitting PCM using an RF carrier.

Many other digital modulation schemes are possible.

The number of quantization levels, \( q = \log_2 n \), controls the quantization error, assuming \( m(t) \) lies within the full-scale range of the quantizer.

Increasing \( q \) reduces the quantization error, but also increases the transmission bandwidth.

The error between \( m(kT_s) \) and the quantized value \( Q[m(kT_s)] \), denoted \( e(n) \), is the quantization error.

If \( n = 16 \), for example, the ratio of signal power in the samples of \( m(t) \), to noise power in \( e(n) \), is about 95 dB (assuming \( m(t) \) stays within the quantizer dynamic range).

---

**Example 3.11: Compact Disk Digital Audio**

- CD audio quality audio is obtained by sampling a stereo source at 44.1 kHz.
- PCM digitizing produces 16 bits per sample per L/R audio channel.
The source bit rate is thus $2 \times 16 \times 44.1\text{kps} = 1.4112\text{ Msps}$.

Data framing and error protection bits are added to bring the total bit count per frame to 588 bits and a serial bit rate of 4.3218 Mbps.

## 3.6 Multiplexing

- It is quite common to have multiple information sources located at the same point within a communication system.

- To simultaneously transmit these signals we need to use some form of multiplexing.
There is more than one form of multiplexing available to the communications engineer.

In this chapter we consider time-division multiplexing, while in Chapter 4 frequency division multiplexing is described.

3.6.1 Time-Division Multiplexing (TDM)

- Time division multiplexing can be applied to sampled analog signals directly or accomplished at the bit level.

- We assume that all sources are sample at or above the Nyquist rate.

- Both schemes are similar in that the bandwidth or data rate of the sources being combined needs to be taken into account to properly maintain real-time information flow from the source to user.

- For message sources with harmonically related bandwidths we can interleave samples such that the wideband sources are sampled more often.

- To begin with consider equal bandwidth sources.
• Suppose that $m_1(t)$ has bandwidth $3W$ and sources $m_2(t), m_3(t),$ and $m_4(t)$ each have bandwidth $W$, we could send the samples as

$$s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_3 \ldots$$

with the commutator rate being $f_s > 2W$ Hz.

• The equivalent transmission bandwidth for multiplexed signals can be obtained as follows

  – Each channel requires greater than $2W_i$ samples/s
  – The total number of samples, $n_s$, over $N$ channels in $T$ s is thus

$$n_s = \sum_{i=1}^{N} 2W_i T$$

  – An equivalent signal channel of bandwidth $B$ would produce $2BT = n_s$ samples in $T$ s, thus the equivalent base-
band signal bandwidth is

\[ B = \sum_{i=1}^{N} W_i \text{ Hz} \]

which is the same minimum bandwidth required for FDM using SSB

- Pure digital multiplexing behaves similarly to analog multiplexing, except now the number of bits per sample, which takes into account the sample precision, must be included

- In the earlier PCM example for CD audio this was taken into account when we said that left and right audio channels each sampled at 44.1 kbps with 16-bit quantizers, multiplex up to

\[ 2 \times 16 \times 44,100 = 1.4112 \text{ Mps} \]
Example 3.12: Digital Telephone System

- The *North American* digital TDM hierarchy is based on a single voice signal sampled at 8000 samples per second using a 7-bit quantizer plus one signaling bit.

- The serial bit-rate per voice channel is 64 kbps.

North American Digital TDM Hierarchy:

<table>
<thead>
<tr>
<th>Digital Signal</th>
<th>Bit Rate R (Mb/s)</th>
<th>PCM VF Channels</th>
<th>Transmission Media Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sys. Number</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DS-0</td>
<td>0.064</td>
<td>1</td>
<td>Wire pairs</td>
</tr>
<tr>
<td>T1</td>
<td>1.544</td>
<td>24</td>
<td>Wire pairs</td>
</tr>
<tr>
<td>T1C</td>
<td>3.152</td>
<td>48</td>
<td>Wire pairs</td>
</tr>
<tr>
<td>T2</td>
<td>6.312</td>
<td>96</td>
<td>Wire pairs</td>
</tr>
<tr>
<td>T3</td>
<td>44.736</td>
<td>672</td>
<td>Coax, radio, fiber</td>
</tr>
<tr>
<td>DS-3</td>
<td>90.254</td>
<td>1344</td>
<td>Radio, fiber</td>
</tr>
<tr>
<td>DS-3C</td>
<td>139.264</td>
<td>2016</td>
<td>Radio, fiber, coax</td>
</tr>
<tr>
<td>T4</td>
<td>274.176</td>
<td>4032</td>
<td>Coax, fiber</td>
</tr>
<tr>
<td>DS-4</td>
<td>432.00</td>
<td>6048</td>
<td>Fiber</td>
</tr>
<tr>
<td>T5</td>
<td>560.160</td>
<td>8064</td>
<td>Coax, fiber</td>
</tr>
</tbody>
</table>

- Consider the T1 channel which contains 24 voice signals.

- Eight total bits are sent per voice channel at a sampling rate of 8000 Hz.

- The 24 channels are multiplexed into a T1 frame with an extra bit for frame synchronization, thus there are $24 \times 8 + 1 = 193$ bits per frame.
• Frame period is $1/8000 = 0.125$ ms, so the serial bit rate is $193 \times 8000 = 1.544$ Mbps

• Four T1 channels are multiplexed into a T2 channel (96 voice channels)

• Seven T2 channels are multiplexed into a T3 channel (672 voice channels)

• Six T3 channels are multiplexed into a T4 channel (4032 voice channels)