$r < 0 \quad r = 0 \quad r > 0$

$f(x,y)$

$x(t, \zeta_0)$

$h_{LP}(t)$

$y(t, \zeta_i), i = 0, 1, \ldots$
Course Introduction/Overview

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1.1 Introduction to Random Signals

- Mathematical models
- Random signals in practice
- Course perspective
- What is this course about?
- The computer simulation project
- Instructor policies
- Course syllabus
1.2 Mathematical Models

- Mathematical models serve as tools in the analysis and design of complex systems

- A mathematical *model* is used to represent, in an approximate way, a physical process or system where measurable quantities are involved

- Typically a computer program is written to evaluate the mathematical model of the system and plot performance curves
  
  – The model can more rapidly answer questions about system performance than building expensive hardware prototypes

- Mathematical models may be developed with differing degrees of fidelity

- A system prototype is ultimately needed, but a computer simulation model may be the first step in this process

- A computer simulation model tries to accurately represent all relevant aspects of the system under study

- Digital signal processing (DSP) often plays an important role in the implementation of the simulation model

- If the system being simulated is to be DSP based itself, the simulation model may share code with the actual hardware prototype

- The mathematical model may employ both deterministic and random signal models
1.2. MATHEMATICAL MODELS

The Mathematical Modeling Process\(^1\)

---

1.3 Engineering Applications

Communications, Computer networks, Decision theory and decision making, Estimation and filtering, Information processing, Power engineering, Quality control, Reliability, Signal detection, Signal and data processing, Stochastic systems, and others.

Relation to Other Subjects

---

1.4 Random Signals in Practice

- A typical application of random signals concepts involves one or more of the following:
  
  - Probability
  - Random variables
  - Random (stochastic) processes

Example 1.1: Modeling with Probability

- Consider a digital communication system with a binary symmetric channel and a coder and decoder

\[ \begin{array}{c}
\text{Input} \\
0 \\
1
\end{array} \quad \begin{array}{c}
\text{Output} \\
1 - \epsilon \\
\epsilon \\
\epsilon
\end{array} \quad \begin{array}{c}
\epsilon \\
1 - \epsilon
\end{array} \]

\[ \epsilon = \text{Error Probability} \]

Communication System with Error Control

A data link with error correction

- The channel introduces bit errors with probability \( P_e \text{ (bit)} = \epsilon \)

- A simple code scheme to combat channel errors is to repetition code the input bits by say sending each bit three times
• The decoder then decides which bit was sent by using a majority vote decision rule.

• The system can tolerate one channel bit error without the decoder making an error.

• A symbol error occurs when either two or three channel bit errors occur.

• The probability of a symbol error is given by

\[ P_e(\text{symbol}) = P(2 \text{ bit errors}) + P(3 \text{ bit errors}) \]

• Assuming bit errors are statistically independent, we can write

\[ P(2 \text{ bit errors}) = \epsilon \cdot \epsilon \cdot (1 - \epsilon) + \epsilon \cdot (1 - \epsilon) \cdot \epsilon + (1 - \epsilon) \cdot \epsilon \cdot \epsilon = 3\epsilon^2(1 - \epsilon) \]

\[ P(3 \text{ errors}) = \epsilon \cdot \epsilon \cdot \epsilon = \epsilon^3 \]

• The symbol error probability is thus

\[ P_e(\text{symbol}) = 3\epsilon^2 - 2\epsilon^3 \]

• Suppose \( P_e(\text{bit}) = \epsilon = 10^{-3} \), then \( P_e(\text{symbol}) = 2.998 \times 10^{-6} \)

• The error probability is reduced by three orders of magnitude, but the coding reduces the throughput by a factor of three.

---

**Example 1.2: Modeling with Random Variables**

• Here we assume that a voltage \( x \) is measured as being only noise, or noise plus signal:

\[
 x = \begin{cases} 
 n, & \text{only noise} \\
 v + n, & \text{noise + signal} 
\end{cases}
\]
1.4. RANDOM SIGNALS IN PRACTICE

- We model \( x \) as a random variable with a \textit{probability density function} dependent upon which hypothesis is present.

\[
\begin{align*}
\text{Area} &= 1 \\
\text{Conditional density function on } x
\end{align*}
\]

- We decide that the hypothesis signal is present if \( x > v_T \), where \( v_T \) is the so-called \textit{decision threshold}.

- The probability of detection is given by

\[
P_D = P(x > v_T | v + n) = \int_{v_T}^{\infty} f_x(x | v + n) \, dx
\]

\[
\begin{align*}
\text{Area corresponding to } P_D
\end{align*}
\]

---

**Example 1.3: Modeling with Random Processes**

- Consider a random or stochastic process of the form

\[
x(t) = A \cos(2\pi f_c t + \theta) + n(t)
\]

which is a sinusoidal carrier plus noise.
In this example the carrier phase $\theta$ is modeled as a random variable and $n(t)$ is modeled as an independent stationary random process.

We may be interested in how to recover the sinusoidal carrier from the noisy signal $x(t)$.

The power spectral density of a random process allows us to see the spectral content of a signal.

The power spectral density of a wide sense stationary random process $x(t)$ is given by the Fourier transform of the autocorrelation function.

In this case the power spectrum is

$$S_{xx}(f) = \frac{A^2}{4} [\delta(f - f_c) + \delta(f + f_c)] + S_{nn}(f)$$

where $S_{nn}(f)$ is the power spectrum of the noise alone.

To recover just the carrier from $x(t)$ we may pass $x(t)$ through a filter.
1.4. RANDOM SIGNALS IN PRACTICE

A \cos(2\pi f_c t + \theta) \rightarrow \begin{array}{c} \text{Filter} \end{array} \rightarrow y(t)

\begin{array}{c}
A \\ n(t)
\end{array}

Signal processing \( x(t) \) to recover just the carrier signal

\begin{array}{c}
\text{Highpass or Bandpass Filter} \\
S_{xx}(f) \\
S_{yy}(f)
\end{array}

Filtering \( x(t) \) to obtain \( y(t) \) with spectrum \( S_{yy}(f) \)

---

Example 1.4: Multiple User Communication Environments

- **Code Division Multiple Access** (CDMA) is utilized in one of the second generation mobile communications systems, i.e., IS-95
- To model the system performance of this system we can use random signals
- A simplified block diagram of an equivalent baseband system is shown below
Detection of user 1 in a $K$ users CDMA system

- The composite received signal can be written as

$$y(t) = \sqrt{P_1/2} d_1(t - \tau_1) c_1(t - \tau_1)$$

$$+ \sum_{k=2}^{K} \sqrt{P_k/2} d_k(t - \tau_k) c_k(t - \tau_k) + n(t)$$

where $P_k$ and $\tau_k$ denote the signal power and propagation delay, respectively, for the $k$-th user, and $n(t)$ is white Gaussian noise.

- The signals $c_k(t)$ are the unique spreading codes associated with each user and the signals $d_k(t)$ are the user data.

- Here we assume $c(t)$ and $d(t)$ take on values of $\pm 1$ over the bit interval.

- Ideally, we choose the spreading codes to be mutually orthogonal, e.g., Walsh codes are used in IS-95.
1.4. RANDOM SIGNALS IN PRACTICE

- Multipath propagation, not modeled here, will prevent perfect orthogonality from being maintained at the receiver.

- Assuming a local despreading code of the form \( c_1(t - \tau_1) \) we have perfect synchronization, and we can write the integrator output for the data bit on \( 0 \leq t \leq T_b \) as

\[
Y = \sqrt{P_1/2} d_1(0) T_b + \sum_{k=2}^{K} \sqrt{P_k/2} T_b d_k(0) \rho_{1k} + N_g
\]

where the first term is the desired signal, the second term constitutes multiple access noise, and the third term is a Gaussian random variable due to the AWGN channel noise.

- The multiple access noise is controlled in part by the aperiodic correlation coefficient from user 1 to user \( k \)

\[
\rho_{1k} = \frac{d_k(-1)}{d_k(0)} \int_0^{\tau_k} c_1(t) c_k(t + T_b - \tau_k) \, dt \\
+ \int_{\tau_k}^{T_b} c_1(t) c_k(t - \tau_k) \, dt
\]

- The decision rule applied to \( Y \) is to declare a +1 is sent if \( Y > 0 \) and a -1 is sent if \( Y < 0 \).

- The exact statistics associated with the random variable \( Y \) are quite complex.

- The multiple access noise can be approximated as a Gaussian random variable and hence result in a rather simple form for the bit error probability (BEP).

- In a paper by Pursley\(^3\) it is shown that

\[
\text{BEP} \approx Q(\sqrt{\text{SNR}})
\]

where
\[
\text{SNR} = \left\{ \frac{K - 1}{3N} + \frac{N_0}{2E_b} \right\}^{-1}
\]

\(K\) is the number of users and \(N\) is the number of spreading code chips per bit, i.e., the processing gain; note here we assume perfect power control so all of the received signal powers are equal.

- The term \(E_b/N_0\) is the ratio of bit energy to noise power spectral density, usually given in dB as \(10 \log_{10}(E_b/N_0)\).
- The function \(Q()\) is the Gaussian \(Q\)-function which is the area under the tail of a zero mean unit variance Gaussian random variable
  \[
  Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du
  \]
- A family of BEP performance curves is shown below.

![CDMA Bit Error Probability](image-url)
Example 1.5: Simulation using Random Processes

- The second generation wireless system *Global System for Mobile Communications* (GSM), uses the Gaussian minimum shift-keying (GMSK) modulation scheme

\[ x_c(t) = \sqrt{2P_c} \cos \left[ 2\pi f_0 t + 2\pi f_d \sum_{n=-\infty}^{\infty} a_n g(t - nT_b) \right] \]

where

\[ g(t) = \frac{1}{2} \left\{ \text{erf} \left[ -\sqrt{\frac{2}{\ln 2}} \pi BT_b \left( \frac{t}{T_b} - \frac{1}{2} \right) \right] \right. \]

\[ + \left. \text{erf} \left[ \sqrt{\frac{2}{\ln 2}} \pi BT_b \left( \frac{t}{T_b} + \frac{1}{2} \right) \right] \right\} \]

- The GMSK shaping factor is \( BT_b = 0.3 \) and the bit rate is \( R_b = \frac{1}{T_b} = 270.833 \) kbps

- We can model the baseband GSM signal as a complex random process

- Suppose we would like to obtain the fraction of GSM signal power contained in an RF bandwidth of \( B \) Hz centered about the carrier frequency

- There is no closed form expression for the power spectrum of a GMSK signal

- A simulation constructed in MATLAB can be used to produce a complex baseband version of the GSM signal
Estimated baseband GSM power spectra

- Using averaged periodogram spectral estimation we can estimate $S_{\text{GMSK}}(f)$ and then find the fractional power in any RF bandwidth, $B$, centered on the carrier

$$P_{\text{fraction}} = \frac{\int_{-B/2}^{B/2} S_{\text{GMSK}}(f) \, df}{\int_{-\infty}^{\infty} S_{\text{GMSK}}(f) \, df}$$

- The integrals become finite sums in the MATLAB calculation
Fractional GSM signal power in a centered $B$ Hz RF bandwidth

- An expected result is that most of the signal power (95%) is contained in a 200 kHz bandwidth, since the GSM channel spacing is 200 kHz

---

**Example 1.6: Separate Queues vs A Common Queue**

A well known queuing theory result$^4$ is that multiple servers, with a common queue for all servers, gives better performance than multiple servers each having their own queue. A chapter on queuing theory is contained near the end of the course text. It is interesting to see probability theory in action modeling a scenario we all deal with in our lives.

CHAPTER 1. COURSE INTRODUCTION/OVERVIEW

Random Arrivals at $\lambda$ per unit time (exponentially distributed)

Customers → One Queue (waiting line) → Departing Customers

Servers

1
2
⋯
m

Rate $\lambda$

$T_s = \text{Average Service Time}$

Common queue versus separate queues for multiple servers

Common Queue Analysis

- The number of servers is defined to be $m$, the mean customer arrival rate is $\lambda$ per unit of time, and the mean customer service time is $T_s$ units of time.

- In the single queue case we let $u = \lambda T_s = \text{traffic intensity}$

- Let $\rho = u/m = \text{server utilization}$

- For stability we must have $u < m$ and $\rho < 1$
• As a customer we are usually interested in the average time in the queue, which is defined as the waiting time plus the service time (Tanner)

\[ T_{CQ}^Q = T_w + T_s = \frac{E_c(m, u)T_s}{m(1 - \rho)} + T_s \]

\[ = \frac{[E_c(m, u) + m(1 - \rho)]T_s}{m(1 - \rho)} \]

where

\[ E_c(m, u) = \frac{u^m / m!}{u^m / m! + (1 - \rho) \sum_{k=0}^{m-1} u^k / k!} \]

is known as the Erlang-C formula

• To keep this problem in terms of normalized time units, we will plot \( T_Q/T_s \) versus the traffic intensity \( u = \lambda T_s \)

• The normalized queuing time is

\[ \frac{T_{CQ}^Q}{T_s} = \frac{E_c(m, u) + (m - u)}{m - u} = \frac{E_c(m, u)}{m - u} + 1 \]

Separate Queue Analysis

• Since the customers randomly choose a queue, arrival rate into each queue is just \( \lambda / m \)

• The server utilization is \( \rho = (\lambda / m) \cdot T_s \) which is the same as the single queue case

• The average queuing time is (Tanner)

\[ T_{SQ}^Q = \frac{T_s}{1 - \rho} = \frac{T_s}{1 - \frac{\lambda T_s}{m}} = \frac{mT_s}{m - \lambda T_s} \]
• The normalized queuing time is

\[
\frac{T^\text{SQ}_Q}{T_s} = \frac{m}{m - \lambda T_s} = \frac{m}{m - u}
\]

• Create the Erlang-C function in MATLAB:

```matlab
function Ec = erlang_c(m,u);
% Ec = erlangc(m,u)
% % The Erlang-C formula
% % Mark Wickert 2001

s = zeros(size(u));
for k=0,
    s = s + u.^k/factorial(k);
end
Ec = (u.^m)/factorial(m)./(u.^m/factorial(m) + (1 - u/m).*s);
```

• We will plot \( T^\text{SQ}_Q / T_s \) versus \( u = \lambda T_s \) for \( m = 2 \) and 4

```matlab
>> % m = 2 case
>> u = 0:.05:1.9;
>> m = 2;
>> Tqsq = m./(m - u);
>> Tqcq = erlang_c(m,u)./(m-u) + 1;
>> % m = 4 case
>> u = 0:.05:3.9;
>> m = 4;
>> Tqsq = m./(m - u);
>> Tqcq = erlang_c(m,u)./(m-u) + 1;
```

• The plots are shown below:
1.4. RANDOM SIGNALS IN PRACTICE

Queuing time of common queue and separate queues for $m = 2$ servers

Queuing time of common queue and separate queues for $m = 4$ servers
1.5 Course Perspective in the Comm/DSP Area of ECE
1.6 What is this course about?

- The text chosen for this course teaches the theory and application of probability, random variables, and random processes.
- The Papoulis text is a *classic* for a graduate electrical engineering course on random signals.
- For this course we are using the fourth edition of the text.
- The assumed background is a previous course in at least probability and random variables at the undergraduate level (e.g., ECE 3610) and linear systems (e.g., ECE 3510).
- The course begins with probability theory, quickly moves to random variables of one, two, and $n$-dimensions, and then the study of random processes.
- Working the assigned homework problems taken from the text and other sources is essential to surviving the course.
1.7 The Role of Computer Analysis/Simulation Tools

• In working homework problems pencil and paper type solutions are mostly all that is needed

• Occasionally an analytical expression may need to be plotted

• Simple simulations can be useful in enhancing your understanding of mathematical concepts

• A computer project which involves both mathematical modeling and Monte-Carlo simulation will be assigned later in the semester

• The use of MATLAB for computer work is encouraged since it is fast and efficient at evaluating mathematical models and running Monte-Carlo system simulations
1.8 Instructor Policies

- Homework papers are due at the start of class
- If business travel or similar activities prevent you from attending class and turning in your homework, please inform me beforehand
- Grading is done on a straight 90, 80, 70, ... scale with curving below these thresholds if needed
- Homework solutions will be placed on the course Web site in PDF format with security password required
- Old exams will be placed on the Web site prior to the hour exams, also with a security password required
CHAPTER 1. COURSE INTRODUCTION/OVERVIEW

1.9 Course Syllabus

ECE 5610/4610
Random Signals
Fall Semester 2004

Instructor: Mark Wickert  
Office: EB-226  
Phone: 262-3500  
Fax: 262-3589

wickert@eas.uccs.edu  
http://eceweb.uccs.edu/wickert/ece5610/

Office Hrs: Tue. 10:45 am–12:00 pm, 3:15–4:15 pm and after 7:05 pm as needed, others by appointment. Note: These hours may be adjusted if needed.


Optional Software: MATLAB Student Version with Simulink with MATLAB 7.x, Simulink 5.x, and Symbolic Math Toolbox (no matrix size limits). An interactive numerical analysis, data analysis, and graphics package for Windows/Linux/Mac OSX $99.95. The Signal Processing toolbox may be useful at $29.95. Order from www.mathworks.com/student. Note: The ECE PC Lab has the full version of MATLAB and Simulink for windows (ver. 7.0) with many toolboxes.

Grading: 1.) Graded homework assignments and computer projects worth 40%.  
2.) Two “Hour” exams at 15% each, 30% total.  
3.) Final exam worth 30%.

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Selected Topics Chosen From the Following:

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