Review of Transmission Line Theory

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Review of Transmission Line Theory

- Transmission lines and waveguides are used to transport electromagnetic energy at microwave frequencies from one point in a system to another.
- The desirable features of a transmission line or waveguide are:
  - Single-mode propagation over a wide band of frequencies
  - Small attenuation
- The transmission line structures of primary interest for this course are those for which the dominant mode of propagation is a transverse electromagnetic (TEM) wave.
- Recall that for TEM waves the components of electric and magnetic fields in the direction of wave propagation are zero.
- We wish to consider transmission lines which consist of two or more parallel conductors which have axial uniformity.
- That is to say their cross-sectional shape and electrical properties do not vary along the axis of propagation.
• The TEM wave solution for axially uniform transmission lines can be obtained using:
  – Field Analysis - (obtain electric and magnetic field waves analogous to uniform plane waves)
  – Distributed-Circuit Analysis - (obtain voltage and current waves)

• Distributed circuit analysis will be at the forefront of all analysis in this course, in particular consider Pozar\(^1\), “Modern microwave engineering involves predominantly distributed circuit analysis and design, in contrast to the waveguide and field theory orientation of earlier generations”

Parameters:

$L$ – series inductance per unit length due to energy storage in the magnetic field
$C$ – shunt capacitance per unit length due to energy storage in the electric field
$R$ – series resistance per unit length due to power loss in the conductors
$G$ – shunt conductance per unit length due to power loss in the dielectric. (i.e. $\varepsilon = \varepsilon' - j\varepsilon''$, $\varepsilon'' \neq 0$)

- Using KCL, KVL and letting $\Delta z \to 0$ it can be shown that

$$\frac{-\partial v(z, t)}{\partial z} = Ri(z, t) + L \frac{\partial i(z, t)}{\partial t} \quad (1.1)$$

and

$$\frac{-\partial i(z, t)}{\partial z} = Gv(z, t) + C \frac{\partial v(z, t)}{\partial t} \quad (1.2)$$
• For now assume the line is lossless, that is $R = 0$ and $G = 0$, so we have:

$$\begin{align*}
\frac{-\partial v}{\partial z} &= L \frac{\partial i}{\partial t} \\
\frac{-\partial i}{\partial z} &= C \frac{\partial v}{\partial t} \quad (1.3)
\end{align*}$$

• Now differentiate the first equation with respect to $z$ and the second with respect to time $t$

$$\begin{align*}
\frac{\partial^2 v}{\partial z^2} &= -L \frac{\partial^2 i}{\partial t \partial z} \\
\frac{\partial^2 i}{\partial z \partial t} &= C \frac{\partial^2 v}{\partial t^2} \quad (1.4)
\end{align*}$$

• Combine the two resulting equations to get

$$\frac{\partial^2 v}{\partial z^2} = LC \frac{\partial^2 v}{\partial t^2} \quad \text{(voltage eqn.)} \quad (1.5)$$

similarly obtain

$$\frac{\partial^2 i}{\partial z^2} = LC \frac{\partial^2 i}{\partial t^2} \quad \text{(current eqn.)} \quad (1.6)$$

• These are in the form of the classical one-dimensional wave equation, often seen in the form

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 y}{\partial t^2} \quad (1.7)$$

where $v_p$ has dimension and significance of velocity
• A well known solution to the wave equation is

\[ y = y^+(t - \frac{z}{v_p}) + y^-(t + \frac{z}{v_p}) \]  

(1.8)

- \( y^+ \) propagates in the positive \( z \) direction
- \( y^- \) propagates in the negative \( z \) direction

• This solution can be checked by noting that

\[
\frac{\partial y^\pm}{\partial z} = \frac{\partial y^\pm}{\partial x} \frac{\partial x}{\partial z} = \pm 1 \frac{\partial y^\pm}{\partial x} , \quad x = t \pm \frac{z}{v_p} 
\]

(1.9)

• A solution for the voltage wave equation is thus

\[ v(z, t) = v^+(t - \frac{z}{v_p}) + v^-(t + \frac{z}{v_p}) \]  

(1.10)

• The current equation can be written in a similar manner, but it can also be written in terms of \( V^+ \) and \( V^- \) since

\[
-\frac{\partial v}{\partial z} = L \frac{\partial i}{\partial t} 
\]

(1.11)

or

\[
-\left[ \frac{\partial v^+}{\partial z} + \frac{\partial v^-}{\partial z} \right] = L \frac{\partial i}{\partial t} 
\]

(1.12)

Letting \( x = t \pm z/v_p \) and using the chain rule the above equation becomes

\[
-\left[ \frac{-1}{v_p} \frac{\partial v^+}{\partial x} + \frac{1}{v_p} \frac{\partial v^-}{\partial x} \right] = L \frac{\partial i}{\partial t} 
\]

(1.13)
• This implies that

\[ i(z, t) = \frac{1}{L v_p} \left[ v^+ \left( t - \frac{z}{v_p} \right) - v^- \left( t + \frac{z}{v_p} \right) \right] \] (1.14)

or

\[ i(z, t) = \frac{1}{Z_0} \left[ v^+ \left( t - \frac{z}{v_p} \right) - v^- \left( t + \frac{z}{v_p} \right) \right] \] (1.15)

where

\[ v_p = \frac{1}{\sqrt{LC}}, \text{ (velocity of propagation)} \] (1.16)

\[ Z_0 = \sqrt{\frac{L}{C}}, \text{ (characteristic impedance)} \]

• At this point the general lossless line solution is incomplete. The functions \( v^+ \) and \( v^- \) are unknown, but must satisfy the boundary conditions imposed by a specific problem.

• The time domain solution for a lossless line, in particular the analysis of transients, can most effectively be handled by using Laplace transforms.

• If the source and load impedances are pure resistances and the source voltage consists of step functions or rectangular pulses, then time domain analysis is most convenient.

• In the following we will first consider resistive load and source impedances, later the analysis will be extended to complex impedance loads using Laplace transforms.
• The Laplace transform technique will in theory allow for a generalized time-domain analysis of transmission lines

• In Section 2 sinusoidal steady-state analysis will be introduced. This approach offers greatly reduced analysis complexity

**Transient Analysis with Resistive Loads**

**Infinite Length Line**

Assume that the line is initially uncharged, i.e. for $z \geq 0$ and $t \leq 0$

$$v(z, t) = v^+(t - \frac{z}{v_p}) + v^-(t + \frac{z}{v_p}) = 0 \quad (1.17)$$

and

$$i(z, t) = \frac{1}{Z_0} \left[ v^+(t - \frac{z}{v_p}) - v^-(t + \frac{z}{v_p}) \right] = 0 \quad (1.18)$$

the above equations imply that

$$v^+(t - \frac{z}{v_p}) = 0 \text{ for } t - z/v_p < 0 \quad (1.19)$$
and

\[ v(t + z/v_p) = 0 \quad \text{for all } t \]  \hspace{1cm} (1.20)

\textbf{Note:} For the given initial conditions only \( v^+(t - z/v_p) \) can exist on the line.

• We thus conclude that

\[
\begin{aligned}
    v(z, t) &= v^+(t - z/v_p) \\
i(z, t) &= \frac{1}{Z_0} v^+(t - z/v_p)
\end{aligned}
\]

\hspace{1cm} \text{for all } t - z/v_p > 0 \hspace{1cm} (1.21)

• Suppose that at \( t = 0^+ \) a voltage source \( v_g(t) \) is applied through a source resistance \( R_g \), at \( z = 0 \)

• Apply Ohm's law at \( z = 0 \) for \( t > 0 \) and we obtain

\[ v_g(t) - v(0, t) = i(0, t)R_g \]  \hspace{1cm} (1.22)

or

\[ v_g(t) - v^+(t) = \frac{R_g}{Z_g} v^+(t) \]  \hspace{1cm} (1.23)

which implies

\[ v^+(t) = \frac{Z_0}{Z_0 + R_g} v_g(t) \]  \hspace{1cm} (1.24)
• The final result is that under the infinite line length assumption for any \( z \) we can write

\[
v(z, t) = \frac{Z_0}{Z_0 + R_g} v_g \left( t - \frac{z}{v_p} \right)
\]

\[(1.25)\]

\[
i(t, z) = \frac{1}{Z_0 + R_g} v_g \left( t - \frac{z}{v_p} \right)
\]

\[(1.26)\]

– **Note**: That the infinite length of line appears as a voltage divider to the source

– Voltages and currents along the line appear as replicas of the input values except for the time delay \( z/v_p \)

**Terminated Line**

![Terminated line circuit diagram](image)

**Figure 1.4**: Terminated line circuit diagram

• **Note**: As a matter of convenience the reference point \( z = 0 \) has been shifted to the load end of the line

• Suppose that a wave traveling in the \( z^+ \) direction is incident upon the load, \( R_L \), at \( z = 0 \)

• Thus,

\[
v(z, t) = v^+ \left( t - \frac{z}{v_p} \right) \quad \text{and} \quad i(z, t) = \frac{1}{Z_0} v^+ \left( t - \frac{z}{v_p} \right)
\]

\[(1.27)\]
• By applying Ohm’s law at the load, we must have
\[ v(0, t) = R_L i(0, t) \] (1.28)

• This condition cannot, in general, be met by the incident wave alone since
\[ v(z, t) = Z_0 i(z, t) \] (1.29)

• Since the line was initially discharged, it is reasonable to assume that a fraction, \( \Gamma_L \), of the incident wave is reflected from the load resistance, i.e.,
\[ v^-(t) = \Gamma_L v^+(t) \] (1.30)

• The load voltage is now
\[ v(0, t) = v^+(t) + v^-(t) = (1 + \Gamma_L)v^+(t) \] (1.31)

• Similarly the load current is
\[ i(0, t) = \frac{v^+(t)}{Z_0} - \frac{v^-(t)}{Z_0} = \frac{(1 + \Gamma_L)v^+(t)}{Z_0} \] (1.32)

• To satisfy Kirchoff’s laws,
\[ \frac{\text{Net load voltage}}{\text{Net load current}} = R_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \] (1.33)

or the more familiar result
\[ \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \] (1.34)
- **Note:** $\Gamma_L$ is real by assumption
- **Note:** To terminate the line without reflection use $R_L = Z_0$.

**General Transmission Line Problem**

![Circuit Diagram]

**Figure 1.5:** Circuit for general transmission line problem

- From earlier analysis, we know that for $0 \leq t \leq l/v_p = T_l$,

  $v(z, t) = \frac{Z_0}{Z_0 + R_g} v_g \left( t - \frac{z}{v_p} \right)$

  \[
  \begin{aligned}
  v(z, t) &= \frac{Z_0}{Z_0 + R_g} v_g \left( t - \frac{z}{v_p} \right) \\
  i(z, t) &= \frac{1}{Z_0 + R_g} v_g \left( t - \frac{z}{v_p} \right)
  \end{aligned}
  \]

  $0 \leq t \leq T_l$ \hspace{1cm} (1.35)

- When $t = T_l$ the leading edge of $v_g(t)$ has traveled to the load end of the line ($z = l$)

- Assuming $R_L \neq Z_0$ a reflected wave now returns to the source during the interval $T_l \leq t < 2T_l$

  $v(z, t) = \frac{Z_0}{Z_0 + R_g} v_g \left( t - \frac{z}{v_p} \right) + \frac{Z_0}{Z_0 + R_g} \Gamma_L v_g \left( t - \frac{2l}{v_p} + \frac{z}{v_p} \right)$

  \[
  \begin{aligned}
  v(z, t) &= \frac{Z_0}{Z_0 + R_g} v_g \left( t - \frac{z}{v_p} \right) + \frac{Z_0}{Z_0 + R_g} \Gamma_L v_g \left( t - \frac{2l}{v_p} + \frac{z}{v_p} \right) \\
  &= v^+ (t - z/v_p) + \frac{Z_0}{Z_0 + R_g} \Gamma_L v^- (t + z/v_p)
  \end{aligned}
  \]

  $\Gamma_L = \frac{Z_L - R_L}{Z_L + R_L}$
and

\[
  i(z, t) = \frac{1}{Z_0 + R_g} v_g \left( t - \frac{z}{v_p} \right) - \frac{1}{Z_0 + R_g} \Gamma_L v_g \left( t - \frac{2l}{v_p} + \frac{z}{v_p} \right) \tag{1.37}
\]

- When the load reflected wave \( v^- (t + z/v_p) \) arrives at the source \((z = 0)\), a portion of it will be reflected towards the load provided \( R_g \neq Z_0 \)
- The reflection that takes place is independent of the source voltage
- The wave traveling in the positive \( Z \) direction after \( 2l/v_p = 2T_l \) seconds has elapsed, can be found by applying Ohm's law for \( z = 0 \) and \( t = 2T_l \):

\[
  v_g (2T_l) - v(0, 2T_l) = R_g i(0, 2T_l) \tag{1.38}
\]

- Now substitute

\[
  v(0, 2T_l) = v^+(2T_l) + v^-(2T_l)
  
  i(0, 2T_l) = \frac{1}{Z_0} [v^+(2T_l) - v^-(2T_l)] \tag{1.39}
\]

and solve for \( v^+(2T_l) \)
- The results is

\[
  v^+(2T_l) = v_g(2T_l) \frac{Z_0}{Z_0 + R_g} + v^-(2T_l) \frac{R_g - Z_0}{R_g + Z_0}
  
  = v_g(2T_l) \frac{Z_0}{Z_0 + R_g} + v_g(0) \frac{Z_0}{Z_0 + R_g} \Gamma_L \Gamma_g \tag{1.40}
\]
where \( \Gamma_L \) is the source reflection coefficient defined as

\[
\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0}
\]  

(1.41)

**Note:** \( \Gamma_g \) is real by assumption in this case.

- The wave incident on the load during the interval \( 2T_l \leq t < 3T_l \), thus consists of the original source signal plus a reflected component due to mismatches at both the load and source ends of the line:

\[
v(z, t) = \frac{Z_0}{Z_0 + R_g} \left[ v_g \left( t - \frac{z}{v_p} \right) + \Gamma_L v_g \left( t + \frac{z-2l}{v_p} \right) \right] + \Gamma_g \Gamma_L v_g \left( t - \frac{z+2l}{v_p} \right), 2T_l \leq t < 3T_l
\]

(1.42)

and

\[
i(z, t) = \frac{1}{Z_0 + R_g} \left[ v_g \left( t - \frac{z}{v_p} \right) - \Gamma_L v_g \left( t + \frac{z-2l}{v_p} \right) \right] + \Gamma_g \Gamma_L v_g \left( t - \frac{z+2l}{v_p} \right), 2T_l \leq t < 3T_l
\]

(1.43)

- The process of reflections occurring at both the source and load ends continues in such a way that in general over the \( n^{th} \) time interval, \((n-1)T_l \leq t \leq nT_l\), the \( v(z, t) \) and \( i(z, t) \) solutions each require \( n \) terms involving \( v_g(t) \)
Example: Consider the following circuit.

Here we have

\[
\frac{Z_0}{Z_0 + R_g} = \frac{1}{4}, \quad \Gamma_L = -1, \quad \Gamma_g = \frac{1}{2}, \quad T_l = 2\mu s \quad (1.44)
\]

a) Find \( V(0, t) \) and \( I(0, t) \) for \( T_p = 1\mu s \)
b) Find \( V(0, t) \) and \( I(0, t) \) for \( T_p = 6\mu s \)

• For both parts (a) and (b) the basic circuit operation is as follows:
  i) An 8v pulse will propagate toward the load, reaching the load in 2 \( \mu s \).
  ii) A -8v pulse will be reflected from the load, requiring 2 \( \mu s \) to reach the source.
  iii) A -4v pulse will be reflected at the source. It will take 2 \( \mu s \) to reach the load.
  iv) The process continues.

• The +z and -z direction propagating pulses can be displayed on a distance-time plot or bounce diagram (see Figure 1.7)
– In the bounce diagram the boundaries at $z = 0$ and $z = 400$ m are represented as surfaces with reflection coefficient of $1/2$ and -1 respectively.

– To obtain the voltage waveform at say $z = 380$ m as a function of time, you sum the contributions from the various $+z$ and $-z$ traveling waves.

– For pulses that are short in comparison with the one-way delay time of the line, only at most two wave terms need to be included at a time.

– For long pulses (in the limit say a step function) all wave terms need to be included.

\[\text{Figure 1.7: Bounce diagram showing pulse propagation}\]
Review of Transmission Line Theory

a) \( T_p = 1 \mu s \): at \( t = 4 \mu s \) we have
\[
\nu(0, 4 \mu s) = -4 + (-8) = -12v.
\]
\[
i(0, 4 \mu s) = \frac{1}{50}[-4 - (-8)] = 0.08A.
\]

b) \( T_p = 6 \mu s \): at \( t = 4 \mu s \) we have
\[
\nu(0, 4 \mu s) = (8 + (-4)) + (-8) = -4v.
\]
\[
i(0, 4 \mu s) = \frac{1}{50}[(8 + (-4)) - (-8)] = 0.24A.
\]

• We can simulate this result using the Agilent Advanced Design System (ADS) software
• In this example we will use ideal transmission line elements

\[\text{Figure 1.8: Circuit schematic (Example1.dsn)}\]
- The source end voltage, $v(0, t)$

- The source end current entering the line, $i(0, t)$

- Modify the schematic by increasing the pulse width to 6 $\mu$s
- The source end voltage, $v(0, t)$

- The source end current entering the line, $i(0, t)$
Transient Analysis using Laplace Transforms

![Lossless line with arbitrary terminations](image)

**Figure 1.9:** Lossless line with arbitrary terminations

- In the time domain we know that for a lossless line
  
  \[ v(z, t) = v^+(t - \frac{z}{v_p}) + v^-\left(t + \frac{z}{v_p}\right) \]  
  \[ (1.45) \]

  and

  \[ i(z, t) = \frac{1}{Z_0} \left[ v^+(t - \frac{z}{v_p}) - v^-\left(t + \frac{z}{v_p}\right) \right] \]  
  \[ (1.46) \]

  where \( v^+(t - \frac{z}{v_p}) \) and \( v^-\left(t + \frac{z}{v_p}\right) \) are determined by the boundary conditions imposed by the source and load

- Laplace transform each side of the above equations with respect to the time variable, using the time shift theorem which is given by

  \[ \mathcal{L}\{f(t - c)\} = F(s)e^{-sc} \]  
  \[ (1.47) \]

  where \( F(s) \) is the laplace transform of \( f(t) \)

- The result is

  \[ v(z, s) = v^+(s)e^{-sz/v_p} + v^-\left(s\right)e^{sz/v_p} \]  
  \[ (1.48) \]
where \( v^+(s) = \mathcal{L}\{v^+(t)\} \) and \( v^-(s) = \mathcal{L}\{v^-(t)\} \)

**Case 1: Matched Source**

- For the special case of \( Z_g(s) = Z_0 \), the source is matched to the transmission line which eliminates multiple reflections
- Thus, we can write
  \[
  V^+(s) = V_g(s) \frac{Z_0}{Z_0 + Z_0} = \frac{1}{2} V_g(s) \tag{1.50}
  \]
  at \( z = l \), the incident wave is reflected with the reflection coefficient
  \[
  \Gamma_L(s) = \frac{Z_L(s) - Z_0}{Z_L(s) + Z_0} \tag{1.51}
  \]
  so,
  \[
  V(l, s) = V^+(s)e^{-sl/v_p}[1 + \Gamma_L(s)] \tag{1.52}
  \]
  which implies that
  \[
  V^-(s) = \Gamma_L(s)e^{-s2l/v_p}V^+(s) \tag{1.53}
  \]
- Finally, for \( 0 \leq z \leq l \) we can write
  \[
  V(z, s) = \frac{1}{2} V_g(s)[e^{-sz/v_p} + \Gamma_L(s)e^{-s(2l-z)/v_p}] \\
  = \frac{1}{2} V_g(s)e^{-sz/v_p} + \frac{1}{2}\Gamma_L(s)e^{-s2l/v_p}V_g(s)e^{sz/v_p} \tag{1.54}
  \]
• To obtain \( V(z, t) \) inverse transform:

\[
v(z, t) = \frac{1}{2} v_g \left( t - \frac{z}{v_p} \right) + L^{-1} \left[ \Gamma_L(s) V_g(s) \right] \bigg|_{t \rightarrow t + (z - 2l)/v_p} \tag{1.55}
\]

**Example:**

• Let \( v_g(t) = v_0 u(t) \) and \( Z_L \) be a parallel RC connection

\[
\begin{align*}
Z_L &= R \left\| \begin{array}{c} L \\ & C \
\end{array} \right. \\
\end{align*}
\]

**Figure 1.10:** Parallel RC circuit

Find: \( v(z, t) \)

• To begin with in the \( s \)-domain we can write

\[
Z_L(s) = \frac{1}{R + \frac{1}{Cs}} = \frac{R}{1 + RCs}
\tag{1.56}
\]

and

\[
\Gamma_L(s) = \frac{Z_L(s) - Z_0}{Z_L(s) + Z_0} = \frac{R}{1 + RCs} - \frac{Z_0}{1 + RCs + Z_0}
\tag{1.57}
\]

\[
= \frac{R - Z_0}{RCZ_0} - \frac{s}{RCZ_0 + s} = \frac{b - s}{a + s}, \quad a = \frac{R + Z_0}{RCZ_0}, \quad b = \frac{R - Z_0}{RCZ_0}
\]
• Now since \( V_g(s) = v_0 \mathcal{L}\{u(t)\} = v_0 / s \)

\[
V(z, s) = \frac{v_0}{2s} \left[ e^{-sz/v_p} + \frac{b - s}{a + s} e^{-s(2l-z)/v_p} \right] \tag{1.58}
\]

• To inverse transform first apply partial fractions to

\[
\frac{b - s}{s(a + s)} = \frac{K_1}{s} + \frac{K_2}{s + a} \tag{1.59}
\]

Clearly,

\[
K_1 = \frac{b}{a} = \frac{R - Z_0}{R + Z_0} \quad K_2 = \frac{-(a + b)}{a} = \frac{-2R}{R + Z_0} \tag{1.60}
\]

so

\[
V(z, s) = \frac{v_0}{2s} \left[ \frac{1}{s} e^{-sz/v_p} \right.
\]

\[
+ \left\{ \frac{R - Z_0}{R + Z_0} \cdot \frac{1}{s} - \frac{2R}{R + Z_0} \cdot \frac{1}{s + a} \right\} e^{-s(2l-z)/v_p} \right] \tag{1.61}
\]

and

\[
v(z, t) = L^{-1}\{V(z, s)\} = \frac{v_0}{2} \left[ u\left(t - \frac{z}{v_p}\right) + \frac{R - Z_0}{R + Z_0} \right.
\]

\[
- \left( \frac{t - \frac{2l-z}{v_p}}{RCZ_0} \right) R + Z_0 \right]\left\{ u\left(t - \frac{2l-z}{v_p}\right) \right] \tag{1.62}
\]
• As a special case consider \( z = l \)

\[
v(z, t) = \frac{v_0}{2} \left[ 1 + \frac{R - Z_0}{R + Z_0} - \frac{2R}{R + Z_0} e^{-\left(\frac{t - l}{v_p}\right)^{R + Z_0}} \right] u\left( t - \frac{l}{v_p} \right)
\]

(1.63)

\[
v_0 \frac{R}{R + Z_0} \left[ 1 - e^{-\left(\frac{t - l}{v_p}\right)^{R + Z_0}} \right] u\left( t - \frac{l}{v_p} \right)
\]

\[
\text{Time constant } = \frac{RCZ_0}{R + Z_0}
\]

**Figure 1.11:** Sketch of the general \( v(l, t) \) waveform

**Example:** The results of an ADS simulation when using \( v_0 = 2v \), \( Z_0 = R = 50 \) ohms, \( C = 50 \) pf, and \( T_l = 5 \) ns is shown below

**Figure 1.12:** ADS Circuit diagram with nodes numbered

• The ADS schematic is shown below
Next we plot the source and load end waveforms
Case 2: Mismatched Source Impedance

- For the general case where $Z_g(s) \neq Z_0$ to reflections occur at the source end of the line as well as at the load end.
- The expression for $V(z, s)$ now will consist of an infinite number of terms as shown below:

\[
V(z, s) = V_g(s) \frac{Z_0}{Z_0 + Z_g(s)} \left[ e^{-sz/\nu_p} + \Gamma_L(s)e^{-s(2l-z)/\nu_p} \right.
\]
\[
+ \Gamma_L(s)\Gamma_g(s)e^{-s(2l+z)/\nu_p}
\]
\[
+ \Gamma_L(s)\Gamma_g(s)\Gamma_L(s)e^{-s(4l-z)/\nu_p}
\]
\[
+ \Gamma_L(s)\Gamma_g(s)\Gamma_L(s)\Gamma_g(s)e^{-s(4l+z)/\nu_p} + \cdots \]

where

\[
\Gamma_L(s) = \frac{Z_L(s) - Z_0}{Z_L(s) + Z_0} \quad \text{and} \quad \Gamma_g(s) = \frac{Z_g(s) - Z_0}{Z_g(s) + Z_0}
\]

- In terms of $+z$ and $-z$ propagating waves we can write

\[
V(z, s) = \frac{V_g(s)Z_0}{Z_0 + Z_g(s)} \left[ e^{-sz/\nu_p} \sum_{n=0}^{\infty} \Gamma_L^n(s)\Gamma_g^n(s)e^{-s(2n)l/\nu_p} \right.
\]
\[
+ e^{sz/\nu_p} \sum_{n=0}^{\infty} \Gamma_L^{n+1}(s)\Gamma_g^n(s)e^{-s(2n+2)l/\nu_p} \left. \right] \quad (1.66)
\]
**Example:** Consider a circuit with \( v_0 = 2v, Z_0 = 50 \) ohms, \( T_1 = 5 \) ns, \( Z_g = R_g = 100 \) ohms, and \( Z_L \) a parallel RC circuit with \( R = 100 \) ohms and \( C = 20 \) pf, as shown below in Figure 1.13.

![Circuit diagram with Spice nodes indicated](image)

- In the \( s \)-domain the solution is of the form

\[
V(z, s) = \frac{2}{3} \left[ e^{-sz/v_p} \sum_{n=0}^{\infty} \frac{(b-s)^n}{s(s+a)^n} \left( \frac{1}{3} \right)^n e^{-s(2n+2)/v_p} \right] (1.67)
\]

- To inverse transform \( V(z, s) \) note that each series term consists of the product of a constant, a ratio of polynomials in \( s \), and a time shift exponential (i.e. \( e^{-st} \))

- In the time-domain each series term to within a constant is of the form

\[
L^{-1} \left\{ \frac{(b-s)^n}{s(s+a)^n} \right\}_{t \to t - \tau_n}, \quad n = 0, 1, 2, \ldots \quad (1.68)
\]
• A partial fraction expansion of the ratio of polynomials in
(1.68) is
\[
\frac{(b-s)^n}{s(s+a)^n} = \frac{K_1}{s} + \frac{K_{12}}{s+a} + \frac{K_{22}}{(s+a)^2} + \cdots + \frac{K_{2n}}{(s+a)^n}
\] (1.69)
where
\[
K_1 = \frac{b^n}{a^n}
\] (1.70)
and
\[
K_{2k} = \frac{1}{(n-k)!} \frac{d^{(n-k)}}{ds^{(n-k)}} \left[ \frac{(b-s)^n}{s} \right] \bigg|_{s=-a}, \quad k = 1, 2, \ldots, n
\] (1.71)

• To obtain a partial solution for comparison with a Spice sim-
ulation we will solve (1.69) for \( n = 0, 1, \) and 2.
  – Case \( n = 0 \):
    \[
    \frac{1}{s} \Leftrightarrow u(t)
    \] (1.72)
  – Case \( n = 1 \)
    \[
    \frac{b-s}{s(s+a)} = \frac{b/a}{s} - \frac{(a+b)/a}{s+a} \Leftrightarrow \left[ \frac{b}{a} - \frac{a+b}{a} e^{-at} \right] u(t)
    \] (1.73)
\[
\frac{-n = 2}{(b-s)^2} = \frac{b^2/a^2}{s} + \frac{(1-b^2/a^2)}{s+a} - \frac{(b+a)^2/a}{(s+a)^2} \quad (1.74)
\]

\[
\Leftrightarrow \left[ \frac{b^2}{a^2} + \left(1 - \frac{b^2}{a^2}\right)e^{-at} - \frac{(b+a)^2}{a}te^{-at}\right]u(t)
\]

- Using Mathematica the analytical solution valid for \(t\) up to 25 ns was obtained

\[
a = \frac{RL + Z_0}{RL \cdot Z_0} / \{ RL \rightarrow 100, \; Z_0 \rightarrow 50, \; C \rightarrow 20 \times 10^{-12}\};
\]

\[
b = \frac{RL - Z_0}{RL \cdot Z_0} / \{ RL \rightarrow 100, \; Z_0 \rightarrow 50, \; C \rightarrow 20 \times 10^{-12}\};
\]

\[
v_0[t_\_] := \frac{2}{3} UnitStep[t];
\]

\[
v_1[t_\_] := \frac{2}{3} \left(\frac{b}{a} - \frac{a+b}{a} e^{-a \cdot \frac{t}{10^9}}\right) UnitStep[t]
\]

\[
v_2[t_\_] := \frac{2}{3} \left(\frac{b^2}{a^2} + \left(1 - \frac{b^2}{a^2}\right)e^{-a \cdot \frac{t}{10^9}} - \frac{(b+a)^2}{a} \frac{t}{10^9} e^{-a \cdot \frac{t}{10^9}}\right) UnitStep[t]
\]

\[
Plot\left[\{v_0[t] + v_1[t - 10] + \frac{1}{3} v_1[t - 10] + \frac{1}{3} v_2[t - 20] + \frac{1}{9} v_2[t - 20], v_0[t - 5] + \frac{1}{3} v_1[t - 15] + v_1[t - 5] + \frac{1}{3} v_2[t - 15]\}, \{t, 0, 25\},
\right.
\]

\[
PlotRange \rightarrow \{[0, 25], \{-2, 1.2\}\},
\]

\[
PlotStyle \rightarrow AbsoluteThickness[1]
\]
• This required the use of \( +z \) and \( -z \) wave solutions from the series for \( n = 0 \) and 1

• The theoretical voltage waveforms at \( z = 0 \) and \( z = 1 \) are shown below

- A circuit simulation using ADS was also run, the results compare favorably as expected
• Plots of $v(0, t)$ and $v(l, t)$
Sinusoidal Signals on Transmission Lines

Voltage and Current Relationships

- Consider an axially uniform transmission line operating in the TEM or Quasi-TEM mode

\[ I(t,z) \]

\[ V(t,z) \]

\[ z \]

**Figure 1.14:** Voltage and current at \( z \) on an axially uniform line

- Let the sinusoidal voltage and current at \( z \) be given by

\[
v(t, z) = \text{Re}\{V(z)e^{j\omega t}\} = |V(z)| \cos[\omega t + \angle V(z)]
\]

\[
i(t, z) = \text{Re}\{I(z)e^{j\omega t}\} = |I(z)| \cos[\omega t + \angle I(z)]
\]

where the complex phasors \( V \) and \( I \) represent the voltage and current without the time dependence \( e^{j\omega t} \)

- Since a sinusoidal steady-state solution is desired, phasor notation may be used to solve the transmission line equations

- Note that with \( e^{j\omega t} \) suppressed it is implied that

\[
\frac{\partial^n V}{\partial t^n} = (j\omega)^n V \quad (1.76)
\]

- By writing the basic line equations in phasor notation we obtain

\[
\frac{\partial V}{\partial z} = -(R + j\omega L)I \quad (1.77)
\]
and
\[ \frac{\partial I}{\partial z} = -(G + j\omega C)V \] (1.78)

• The voltage wave equation becomes
\[ \frac{\partial^2 V}{\partial z^2} - (RG - \omega^2 LC)V - j\omega(RC + LG)V = 0 \] (1.79)

where \( R \), \( G \), \( L \), and \( C \) are the primary transmission line parameters defined earlier

• The general solution is
\[ V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z} \] (1.80)

where
\[
\gamma = \alpha + j\beta \\
= \sqrt{(R + j\omega L)(G + j\omega C)} \\
= \sqrt{-\omega^2 LC + RG + j\omega(RC + LG)}
\] (1.81)

with \( \alpha \) being the line attenuation constant in nepers per meter and \( \beta \) is the line phase constant in radians per meter

– \( V^+ \) is the constant associated with the wave propagating in the \( +z \) direction and \( V^- \) is the constant associated with the wave propagating in the \(-z\) direction

• The current \( I(z) \) can be shown to have general solution of the form
\[ I(z) = I^+ e^{-\gamma z} + I^- e^{\gamma z} \] (1.82)
• By substituting (1.77) into (1.82) we obtain

\[
I(z) = \left( \frac{-1}{R + j\omega L} \right) \frac{\partial V(z)}{\partial z}
\]

\[
= \frac{-1}{R + j\omega L}(-\gamma V^+ e^{-\gamma z} + \gamma V^- e^{\gamma z})
\]

\[
= \sqrt{\frac{G + j\omega C}{R + j\omega L}} (V^+ e^{-\gamma z} - V^- e^{\gamma z})
\]

• The line characteristic impedance is defined as

\[
Z_0 = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}
\]

which then relates the voltage and current on the line as

\[
Z_0 = \frac{V^+ \text{ also}}{I^+} = \frac{-V^-}{I^-}
\]

• Finally we can write \(I(z)\) as

\[
I(z) = \frac{V^+}{Z_0} e^{-\gamma z} - \frac{V^-}{Z_0} e^{\gamma z}
\]

• In the time domain the steady state solution voltage waveform is

\[
v(z, t) = |V^+| \cos(\omega t - \beta z + \angle V^+) e^{-\alpha z}
\]

\[+ |V^-| \cos(\omega t + \beta z + \angle V^-) e^{\alpha z}
\]
Lossless Line

- Special Case: For an ideal lossless line \( R = G = 0 \), \( \gamma \) and \( Z_0 \) reduce to

\[
\gamma = j\beta = j\omega \sqrt{LC} \quad (\alpha = 0)
\]

\[
Z_0 = \frac{L}{\sqrt{C}}
\]  

(1.88)

- Since \( \beta = \omega \sqrt{LC} \) the phase velocity, \( v_p \), on the line is

\[
v_p = \frac{1}{\sqrt{LC}}
\]  

(1.89)

which allows \( Z_0 \) to also be written as

\[
Z_0 = \frac{1}{v_p C}
\]  

(1.90)

- This result implies that \( Z_0 \) can be obtained by knowing the velocity of propagation in the medium and the capacitance per unit length of the transmission line structure

- In free space

\[
v_p = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \cong 3 \times 10^8 \text{ m/s}
\]  

(1.91)

where \( \varepsilon_0 \) and \( \mu_0 \) are the free space permittivity and permeability respectively

- For a dielectrically filled structure where it is assumed that \( \mu = \mu_0 \), \( \varepsilon = \varepsilon' - j\varepsilon'' \cong \varepsilon_0 \varepsilon_r \) and \( \sigma = 0 \), thus

\[
v_p = \frac{c}{\sqrt{\varepsilon_r}}
\]  

(1.92)
where \( \varepsilon_r \) is the relative permittivity of the medium

- For partially filled transmission line structures such as microstrip, the velocity of propagation can be written as

\[
\nu_p = \frac{c}{\sqrt{\varepsilon_{\text{eff}}}}
\]  

(1.93)

where \( \varepsilon_{\text{eff}} \) is the effective value of relative permittivity. (Note: \( \varepsilon_{\text{eff}} \leq \varepsilon_r \)).

**Low Loss Line**

- Special Case: For most practical transmission line structures the losses are small, which is to say, \( R \ll \omega L \) and \( G \ll \omega C \)
- This low-loss assumption allows the expression for \( \gamma \) to be simplified
- To begin with write

\[
\gamma = j\omega \sqrt{LC} \left[ 1 + \frac{RG + j\omega (RC + LG)}{-\omega^2 LC} \right]^{1/2}
\]  

(1.94)

using the binomial expansion

\[
(1 + x)^{1/2} \approx 1 + \frac{1}{2}x \quad \text{for} \quad |x| \ll 1
\]

- Thus

\[
\gamma \approx j\omega \sqrt{LC} \left[ 1 + \frac{1}{2} \left( \frac{-RG}{\omega^2 LC} - j \omega \left( \frac{R}{L} + \frac{G}{C} \right) \right) \right]
\]  

(1.95)
or

\[
\gamma = \alpha + j\beta \approx \frac{1}{2} \sqrt{LC} \left(\frac{R}{L} + \frac{G}{C}\right) + j\omega \sqrt{LC}
\]  

(1.96)

Note that \( \beta \) did not change from the lossless case

- Using a similar approach on \( Z_0 \) results in

\[
Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{(R + j\omega L)(G - j\omega C)}{G^2 + \omega^2 C^2}}
\]

\[
\approx \sqrt{\frac{RG + j\omega(GL - RC) + \omega^2 LC}{\omega^2 C^2}}
\]

\[
= \sqrt{\frac{L}{C}} \sqrt{\frac{1}{1 + \frac{RG + j\omega(GL - RC)}{\omega^2 LC}}}
\]

\[
= \sqrt{\frac{L}{C}} \left[ 1 + \frac{1}{2} j \left( \frac{G}{\omega C} - \frac{R}{\omega L} \right) \right] \approx \sqrt{\frac{L}{C}}
\]

(1.97)

Note also that under the low-loss assumption \( Z_0 \) is still approximately real
Terminated Lossless Line

First consider the case where the generator or source driving the line is matched to the line characteristic impedance $Z_0$ as shown below in Figure 1.15.

![Figure 1.15: Lossless line terminated with impedance $Z_0$](image)

- For an arbitrary load impedance, $Z_L$, the boundary conditions will require both forward (+z) and backward (-z) propagating waves to exist.
- At $z = 0$, the voltage and current equations with $\alpha = j\beta$ are
  \[ V(0) = V^+ e^{-j\beta(0)} + V^- e^{j\beta(0)} = V_L \]  
  \[ I(0) = \frac{1}{Z_0} V^+ e^{-j\beta(0)} - \frac{1}{Z_0} V^- e^{j\beta(0)} = I_L \]

- By convention, the voltage reflection coefficient at the load is defined as
  \[ \Gamma = \frac{V^-}{V^+} \]
• From the voltage and current equations we can now solve for \( \Gamma \) in terms of \( Z_L \) and \( Z_0 \)

\[
\frac{V(0)}{I(0)} = Z_L = \frac{V^+(1 + \Gamma)}{V^+(1 - \Gamma)/Z_0} = Z_0 \frac{1 + \Gamma}{1 - \Gamma}
\]  

(1.101)

or

\[
\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}
\]

(1.102)

• The voltage transmitted to the load due to the incident voltage wave can be defined in terms of the voltage transmission coefficient \( T \)

\[
V_L = TV^+ = (1 + \Gamma)V^+
\]

(1.103)

so

\[
T = 1 + \Gamma
\]

(1.104)

• At a discontinuity in a transmission line system, such as load \( Z_L \) terminating the line at \( z = 0 \), the ratio of power incident to the power reflected is a quantity of interest

• A common scalar network analyzer measurement is return loss (RL) which is \( 10\log_{10} \) of this power ratio

• The return loss is related to \( |\Gamma| \) in the following way

\[
RL = 10\log_{10}\left(\frac{P_{\text{incident}}}{P_{\text{reflected}}}\right)
\]

(1.105)

where
Sinusoidal Signals on Transmission Lines

\[ P_{\text{incident}} = \frac{1}{2} \text{Re} \{ V^+ (I^*) \} = \frac{1}{2} \frac{|V^+|^2}{Z_0} \]  

\[ P_{\text{reflected}} = \frac{1}{2} \text{Re} \{ V^- (I^*) \} = \frac{1}{2} \frac{|V^-|^2}{Z_0} \]  

(1.106)

• Now

\[ \frac{P_{\text{incident}}}{P_{\text{reflected}}} = \frac{|V^+|^2}{|V^-|^2} = \frac{1}{|\Gamma|^2} \]  

(1.107)

so

\[ RL = -20 \log_{10} |\Gamma| \text{ dB} \]  

(1.108)

• If \( Z_L = Z_0 \) then \( \Gamma = 0 \) and the magnitude of the voltage along the line is just that of the incident wave which is \( |V^+| \)

• In general \( \Gamma \neq 0 \) so

\[ V(z) = V^+ e^{-j\beta z} + \Gamma V^+ e^{j\beta z} \]  

(1.109)

and

\[ |V(z)| = \left| V^+ \right| \left| 1 + \Gamma e^{j2\beta z} \right| \]  

(1.110)

by writing \( \Gamma = \rho e^{j\theta} \) (polar form) then we can write

\[ |V(z)| = \left| V^+ \right| \left| 1 + \rho e^{j(\theta + 2\beta z)} \right| \]  

(1.111)

which inspired the vector diagram of \( |V(z)| \) shown below
• With the aid of the vector diagram shown above, it is clear that $|V(z)|$ takes on maximum and minimum values of

$$
|V(z)|_{\text{max}} = |V^+| |1 + \rho|, \theta + 2\beta z = 2n\pi
$$

$$
|V(z)|_{\text{min}} = |V^+| |1 - \rho|, \theta + 2\beta z = 2n\pi + \pi
$$

where $n$ is an integer

• The variation in $|V(z)|$ is sinusoidal with the distance between maxima and between minima each being $d = \pi/\beta = \lambda/2$

• Recall that $\beta = 2\pi/\lambda$ where $\lambda$ is the wavelength of TEM waves in the medium

• The incident and reflected voltage waves interfere to produce the voltage standing-wave pattern shown below in Figure 1.17
The ratio of $|V(z)|_{\text{max}}$ to $|V(z)|_{\text{min}}$ is defined as the voltage standing-wave ratio (VSWR), or simply SWR

$$\text{VSWR} = \frac{1 + \rho}{1 - \rho} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad (1.113)$$

Comparison of termination characterization parameters

| $|\Gamma|$ | RL   | VSWR  |
|----------|------|-------|
| 0.0      | $+\infty$ | 1.0   |
| 0.1      | +20 dB  | 1.2222|
| 0.3162   | +10 dB  | 1.9520|
| 0.5012   | +6 dB    | 3.0095|
| 0.8913   | +1 dB    | 17.3910|
| 0.9441   | +0.5 dB  | 34.7532|
The analysis of a terminated transmission line up to this point has assumed that the source impedance and line characteristic impedance are equal. This simplifies the analysis of voltage and current along the line since under this assumption the wave reflected at the load is completely absorbed when it arrives at the source.

- To consider the mismatched source case, the impedance seen looking into an arbitrarily terminated transmission line is helpful.

\[
Z_{\text{in}} = \left. \frac{V(z)}{I(z)} \right|_{z=-l} = Z_0 \frac{V^+ e^{j\beta l} + V^- e^{-j\beta l}}{V^+ e^{j\beta l} - V^- e^{-j\beta l}}
\]

(1.114)

Recall that the load reflection coefficient is given by

\[
\Gamma_L = \frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0}
\]

(1.115)
so

\[ Z_{\text{in}} = Z_0 \frac{e^{j\beta l} + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j\beta l}}{e^{j\beta l} - \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j\beta l}} \]  

(1.116)

Finally after rearranging using Euler’s identity for \( \tan(\ ) \) \( Z_{\text{in}} \) reduces to

\[ Z_{\text{in}} = \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \]  

(1.117)

- Two special cases of interest that will be considered later result when the operating frequency is such that \( l = \lambda/2 \) or \( l = \lambda/4 \)
- The input impedances for these cases are

\[ Z_{\text{in}}(l = \lambda/2) = Z_L \]  

\[ Z_{\text{in}}(l = \lambda/4) = \frac{Z_0^2}{Z_L} \]  

(1.118)

- The input impedance of open and short circuit terminated transmission lines will also be of interest

\[ Z_{\text{in}} \Big|_{Z_L \rightarrow 0} = jZ_0 \tan \beta l \]  

\[ Z_{\text{in}} \Big|_{Z_L \rightarrow \infty} = -jZ_0 \cot \beta l \]  

(1.119)
• The reflection coefficient at any \( z = -l \) can also be obtained by noting that

\[
V^+|_{z = -l} = V^+ e^{j\beta l}
\]

\[
V^-|_{z = -l} = V^- e^{-j\beta l}
\]

so

\[
\Gamma(l) = \frac{V^- e^{-j\beta l}}{V^+ e^{j\beta l}} = \Gamma L e^{-j2\beta l}
\]

(1.121)

• The above result will appear later during the discussion of scattering parameters.

**Terminated Lossless Line with Arbitrary Source Impedance**

The circuit of interest is shown below in Figure 1.26

![Terminated line with arbitrary source impedance](image)

**Figure 1.19:** Terminated line with arbitrary source impedance.

• One approach to this problem is to consider all the reflections and re-reflections and then sum the infinite series

• Since the infinite series is geometric and \(|\Gamma| \leq 1\), it is summable in closed form
• A second approach is to use the boundary conditions at the source and load terminations to obtain the steady-state solution directly.

• Using (1.117) for $Z_{in}$ we can write

$$V(z = l) = V_g \frac{Z_{in}}{Z_{in} + Z_g} = V^+ e^{j \beta l} + V^- e^{-j \beta l}$$  \hspace{1cm} (1.122)$$

• Now since $V^- = V^+ \Gamma_L$

$$V_g \frac{Z_{in}}{Z_{in} + Z_g} = V^+ [e^{j \beta l} + \Gamma_L e^{-j \beta l}]$$  \hspace{1cm} (1.123)$$

so

$$V^+ = V_g \frac{Z_{in}}{Z_{in} + Z_g} \frac{e^{-j \beta l}}{1 + \Gamma_L e^{-j 2 \beta l}}$$  \hspace{1cm} (1.124)$$

• Now substitute

$$Z_{in} = Z_0 \frac{1 + \Gamma_L e^{-j 2 \beta l}}{1 - \Gamma_L e^{-j 2 \beta l}}$$  \hspace{1cm} (1.125)$$

into (1.124) to obtain

$$V^+ = V_g \frac{Z_0 e^{-j \beta l}}{[Z_0(1 + \Gamma_L e^{-j 2 \beta l}) + Z_g(1 - \Gamma_L e^{-j 2 \beta l})]}$$  \hspace{1cm} (1.126)$$

$$= V_g \frac{Z_0}{Z_0 + Z_g} \frac{e^{-j \beta l}}{1 - \frac{\Gamma L e^{-j 2 \beta l}}{Z_0 + Z_g}}$$
where $\Gamma_g$ is the reflection coefficient looking towards the generator

- Finally, we can write for any $-l \leq z \leq 0$

$$V(z) = V_g \frac{Z_0}{Z_0 + Z_g} \frac{1 + \Gamma_L e^{j2\beta z}}{1 - \Gamma_g \Gamma_L e^{-j2\beta l}} e^{-j\beta(z + l)}$$

(1.127)

An equation for $I(z)$ may be obtained in a similar manner.

### Source to Load Power Transfer Considerations

- The power delivered to the load via the lossless line is just the input power which is given by

$$P = \frac{1}{2} \text{Re}\{V(-l)I^*(-l)\} = \frac{1}{2} \left\{ V(-l) \frac{V^*(-l)}{Z_{in}} \right\}$$

(1.128)

$$= \frac{1}{2} |V_g|^2 \left| \frac{Z_{in}}{Z_{in} + Z_g} \right|^2 \text{Re}\left\{ \frac{1}{Z_{in}^*} \right\}$$

- Let $Z_{in} = R_{in} + jX_{in}$ and $Z_g = R_g + jX_g$, then

$$P = \frac{1}{2} |V_g|^2 \frac{R_{in}}{(R_{in} + R_g)^2 + (X_{in} + X_g)^2}$$

(1.129)

- As a special case suppose that $Z_L = Z_0$
- Clearly $\Gamma_L = 0$, $Z_{in} = Z_0$, and

$$P = \frac{1}{2} |V_g|^2 \frac{Z_0}{(Z_0 + R_g)^2 + X_g^2}$$

(1.130)
– As another special case suppose that $Z_{\text{in}} = Z_g$
– **Note** that the condition $Z_{\text{in}} = Z_g$ can be satisfied through adjustment of $\beta l$ and $Z_0$

$$P = \frac{1}{2} |V_g|^2 \frac{R_g}{4(R_g^2 + X_g^2)}$$  \hspace{1cm} (1.131)

– How do we obtain *maximum power transfer*?
– For $Z_g$ fixed we know from circuit theory that for maximum power transfer we choose $Z_{\text{in}} = Z_g^*$
– The resulting power delivered is as expected

$$P = \frac{1}{2} |V_g|^2 \frac{1}{4R_g}$$  \hspace{1cm} (1.132)

– Note that if $Z_g$ is real then (1.130) and (1.131) yield the maximum power transfer result

**Terminated Lossy Line**

- The analysis of a terminated lossy line is very similar to the lossless case except now $j\beta$ must be replaced by $\gamma = \alpha + j\beta$
- It will be assumed that $\alpha$ is small enough to imply that $Z_0$ can still be considered real

![Figure 1.20: Terminated lossy line.](image-url)
• To begin with we can immediately write

\[ V(z) = V^+ [e^{-\gamma z} + \Gamma_L e^{\gamma z}] \]

\[ I(z) = \frac{V^+}{Z_0} [e^{-\gamma z} - \Gamma_L e^{\gamma z}] \]  (1.133)

• Additionally \( \Gamma_L \) and \( Z_{in} \) respectively become

\[ \Gamma(l) = \Gamma_L e^{-j2\gamma l} = [\Gamma_L e^{-j2\beta l}] e^{-2\alpha l} \]  (1.134)

and

\[ Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \]  (1.135)

• The power delivered to the line input is

\[ P_{in} = \frac{1}{2} \text{Re} \{ V(-l)I^*(-l) \} \]

\[ = \frac{|V^+|^2}{2Z_0} \left\{ e^{2\alpha l} - \frac{|\Gamma_L|^2}{e^{2\alpha l}} e^{-2\alpha l} + \Gamma_L e^{-j2\beta l} - (\Gamma_L e^{-j2\beta l})^* \right\} \]

\[ = \frac{|V^+|^2}{2Z_0} \left[ 1 - |\Gamma(l)|^2 e^{2\alpha l} \right] \]  (1.136)

• The power actually delivered to the load is

\[ P_L = \frac{1}{2} \text{Re} \{ V(0)I^*(0) \} = \frac{|V^+|^2}{2Z_0} \left[ 1 - |\Gamma_L|^2 \right] \]  (1.137)
• The power lost in the line is

\[ P_{\text{loss}} = P_{\text{in}} - P_L = \frac{|V^+|^2}{2Z_0} \left[ (e^{2\alpha l} - 1) + |\Gamma_L|^2 (1 - e^{-2\alpha l}) \right] \] (1.138)

**Line Attenuation Calculation**

The perturbation method of loss analysis assumes that the fields associated with the lossy line are very similar to the fields of the lossless line.

• The power flow along the line is given by

\[ P(z) = P_0 e^{-2\alpha z} \] (1.139)

where \( P_0 \) is the power input at \( z = 0 \), and \( \alpha \) is the attenuation parameter of interest

• The power loss per unit length is

\[ P_l = -\frac{\partial P}{\partial z} = 2\alpha P_0 e^{-2\alpha z} = 2\alpha P(z) \] (1.140)

• Rearranging we obtain

\[ \alpha = \frac{P_l(z)}{2P(z)} = \frac{P_l(z = 0)}{2P_0} \] (1.141)

• Note that this analysis assumes that \( P_0 \) is known and \( P_l \) can be found from the fields of the lossless line.
Dispersion

• In the first order approximation to $\gamma$ given by (1.96) we found that

$$\beta = \omega \left[ \sqrt{LC} - \frac{RG}{2\omega^2 \sqrt{LC}} \right]$$  \hspace{1cm} (1.142)

• If $\beta$ is not of the form $\beta = a\omega$, then the phase velocity $v_p = \omega / \beta \neq$ constant

• If $v_p$ is a function of frequency then the frequency components associated with a broadband signal will arrive at the load end of the line at different times

• This phenomenon is known as dispersion.

• In linear system theory we would say the system has nonlinear phase

• Fortunately since $RG$ is typically much less than $2\omega^2 \sqrt{LC}$, we see that for low loss lines $\beta$ is very nearly linear in $\omega$

**Example:** Consider the transmission line circuit shown below in Figure 1.21.

![Figure 1.21](image-url)

**Figure 1.21:** Match terminated low loss line.
• We can immediately write that
\[
v_L(\omega) = \frac{1}{2} v_g(\omega) e^{-\alpha l} e^{-j\omega \left[ \sqrt{LC} - \frac{RG}{2\omega^2 \sqrt{LC}} \right]}
\] (1.143)

• The transfer function denoted \( H(\omega) = H(2\pi f) \) is thus given by
\[
H(\omega) = \frac{v_L(\omega)}{v_g(\omega)} = \frac{1}{2} e^{-\alpha l} e^{-j\omega \left[ \sqrt{LC} - \frac{RG}{2\omega^2 \sqrt{LC}} \right]}
\] (1.144)

• The magnitude and phase are
\[
|H(\omega)| = \frac{1}{2} e^{-\omega l}
\]
\[
\angle H(\omega) = -\omega l \sqrt{LC} + \frac{RGl}{2\omega \sqrt{LC}}
\] (1.145)

• For \( RG \neq 0 \) we see that the phase function is composed of linear and nonlinear components

**Distortionless Line**

• It is possible for a lossy line to have a linear phase factor if the line parameters satisfy
\[
\frac{R}{L} = \frac{G}{C}
\] (1.146)

• To demonstrate this insert (1.146) into (1.96)
Clearly, $\beta$ is now a linear function of frequency and the attenuation, $\alpha$, is nearly constant with frequency.

**Note** that the line resistance is usually a weak function of frequency.

**Example:** RG-58/U Coax

- To better illustrate the impact of dispersion, consider the special case of RG-58/U coaxial cable.
- In the Pozar text (and Collin) the transmission parameters of coax are shown to be

\[
\gamma = j\omega \sqrt{LC} \left[ 1 + \frac{R^2}{\omega^2 L^2} - 2j \frac{R}{\omega L} \right]^{1/2} = j\omega \sqrt{LC} \left[ 1 - j \frac{R}{\omega L} \right] = R \sqrt{\frac{C}{L}} + j\omega \sqrt{LC} = \alpha + j\beta
\]  

(1.147)

where $R_s$ is the surface resistivity of the conductor, and is given by

\[
R_s = \sqrt{\frac{\omega \mu}{2\sigma}},
\]  

(1.148)
Sinusoidal Signals on Transmission Lines

where here $\mu = \mu_0 \mu_r = \mu_0$ with $\mu_0 = 4\pi \times 10^{-8}$

- For RG-58/U the dielectric diameter is $2b = 0.116$ in and the dielectric material is polyethylene having $\varepsilon_r = 2.25 \ @ \ 10$ GHz and $\varepsilon'' = 0.004 \varepsilon_0 \varepsilon_r$.

- Since the characteristic impedance is nominally 50 ohms we can determine the inner radius, $a$, by setting

$$Z_0 \approx \sqrt{\mu \frac{L}{C}} = \frac{60}{\sqrt{\varepsilon_r}} \ln \frac{b}{a} = 50,$$

thus $2a = 0.033$ in

- The transmission line parameters are given by:

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a} = \frac{\sqrt{\varepsilon_r} Z_0}{c} = 250 \text{ nH/m}$$

$$C = \frac{2\pi \varepsilon'}{\ln b / a} = \frac{\sqrt{\varepsilon_r}}{c Z_0} = 100 \text{ pf/m} \quad \text{(1.149)}$$

$$R = \frac{R_s}{2\pi} \left( \frac{1}{a} + \frac{1}{b} \right) = 1.273 \times 10^{-4} \times \sqrt{f} \text{ ohms/m}$$

$$G = \frac{2\pi \omega \varepsilon''}{\ln b / a} = \frac{\omega C \varepsilon''}{\varepsilon'} = 2.513 \times 10^{-13} \times f \text{ s/m}$$

- To obtain the frequency response of a one meter section of properly terminated RG-58/U cable, we simply insert the above transmission line parameters into
\[ H(f) = \frac{v_L(f)}{v_g(f)} = \frac{1}{2} e^{-j\gamma l} \]

\[ = \frac{1}{2} \exp \left[ -j2\pi f\sqrt{LC} \sqrt{1 + \frac{RG + j2\pi f(RC + LG)}{-4\pi^2 f^2 LC}} \right] \quad (1.150) \]

making sure to explicitly include the frequency dependence on \( R \) and \( G \), i.e., \( R = R(f) \) and \( G = G(f) \)

- The equivalent impulse response of the system can be obtained by inverse Fourier transformation, i.e.,

\[ h(t) = \mathcal{F}^{-1}\{H(f)\} = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft} df \quad (1.151) \]

- The response of the system to an arbitrary pulse waveform, \( v_g(t) = p(t) \), can be obtained in the transform domain, or by direct convolution

\[ v_L(t) = p(t)*h(t) = \mathcal{F}^{-1}\{P(f)H(f)\} \quad (1.152) \]

where \( P(f) = \mathcal{F}\{p(t)\} \)

- The following results were obtained using Mathematica
  - Magnitude and phase frequency response for \( l = 2 \text{ m} \)
  - Impulse response computed using a 1024 point inverse Fourier transform (IFFT) for \( l = 2, 10, \text{ and } 50 \text{ m} \)
  - Simulated pulse response to a 500 Mb/s return-to-zero (RZ) data pattern of the form ...000101001000... (each pulse is 2ns long)
• Mathematica modeling

\[ L = 250 \times 10^{-9}; \]
\[ CC = 100 \times 10^{-12}; \]
\[ R = 1.273 \times 10^{-4} \sqrt{f}; \]
\[ C = 2.513 \times 10^{-13} f; \]
\[ H = \frac{1}{2} \exp\left[ -j 2 \pi f \sqrt{L CC} \left( 1 + \frac{RG + j 2 \pi f (RC + LG)}{-4 \pi^2 f^2 L CC} \right) \right]; \]

\[ \text{LogLinearPlot}[6.02 + 20 \log[10, \text{Abs}[H]] /. \{ f \rightarrow 2, \{ f, 10^7, 10^{11} \}, \text{PlotRange} \to \{ \{10^7, 10^{11}\}, \{-10, 0\} \}] \]

\[ \text{LogLinearPlot}\left[ \frac{180}{\pi} \text{Arg}\left[ H \exp\left[ j 2 \pi f \sqrt{L CC} \right] \right] \right] /. \{ f \rightarrow 2, \{ f, 10^7, 10^{11} \}, \text{PlotRange} \to \text{All} \]

\[ \text{HT} = \text{Table}\left[ \text{Evaluate}[H /. \{ f \rightarrow k \frac{20 \times 10^9}{1024} + .001, 1 \rightarrow 2 \}], \{k, 0, 1023\} \right]; \]

\[ \text{hTT} = \text{InverseFourier}[\text{HT}, \text{FourierParameters} \rightarrow \{1, 1\}]; \]

\[ \text{ListLinePlot}[\{\text{Re}[\text{hTT}], \text{Im}[\text{hTT}]\}, \text{PlotRange} \to \{\{150, 250\}, \text{All}\}]; \]
• Frequency response

Gain in dB less the 1/2 loss factor of 6 dB
• The phase response and pulse response

- The input is a real signal, but the frequency response is not conjugate symmetric, so the impulse response is complex
- This makes the pulse response contain real and imaginary parts as well
• ADS schematic when using the lossy coax model

• Input and output waveforms with ADS are more realistic, but a warning regarding a complex impulse response is posted