Chapter 9

Continuous-Time Signals and LTI Systems

At the start of the course both continuous and discrete-time signals were introduced. In the world of signals and systems modeling, analysis, and implementation, both discrete-time and continuous-time signals are a reality. We live in an *analog world*, is often said. The follow-on courses to ECE2610, Circuits and Systems I (ECE2205) and Circuits and Systems II (ECE3205) focus on continuous-time signals and systems. In particular circuits based implementation of systems is investigated in great detail. There still remains a lot to discuss about continuous-time signals and systems without the need to consider a circuit implementation. This chapter begins that discussion.

Continuous-Time Signals

- To begin with signals will be classified by their support interval

Two-Sided Infinite-Length Signals

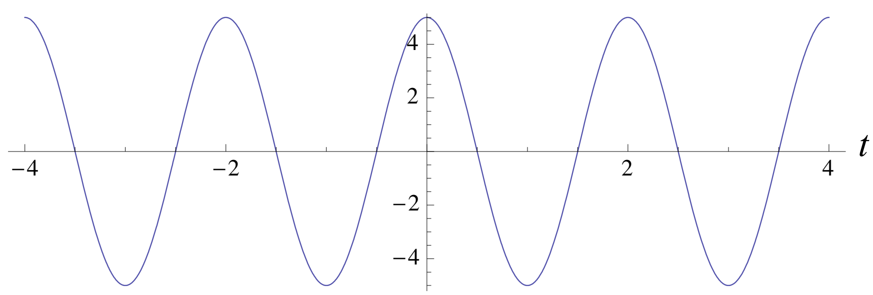
- Sinusoids are a primary example of infinite duration signals, that are also periodic

$$x(t) = A\cos(\omega_0 t + \phi), -\infty < t < \infty$$
$$x(t) = Ae^{j\phi}e^{j\omega_0 t}, -\infty < t < \infty \tag{9.1}$$

- The period for both the real sinusoid and complex sinusoid signals is $T_0 = 2\pi/\omega_0$
- The signal may be any periodic signal, say a pulse train or squarewave
- A two-sided exponential is another example

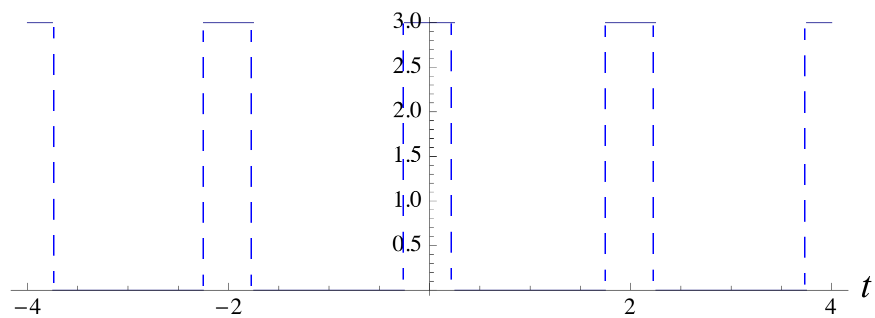
$$x(t) = Ae^{-\beta|t|}, -\infty < t < \infty \tag{9.2}$$

$x(t) = 5\cos\left(2\pi\frac{t}{2}\right)$



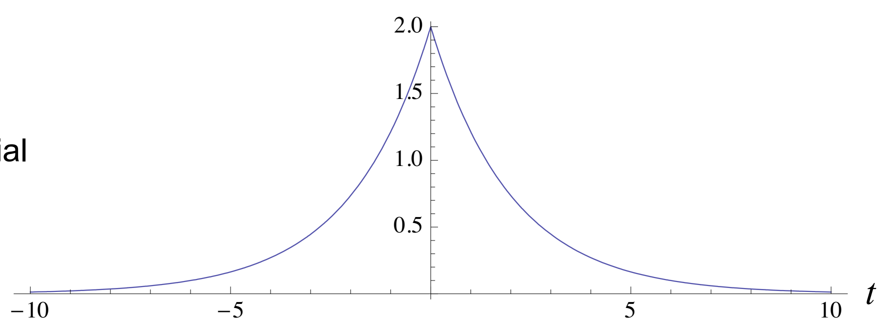
$x(t) =$ Pulse Train

Period = 2s
Pulse Width = 0.5s



$x(t) = 2e^{-|t|/2}$

Two-sided exponential



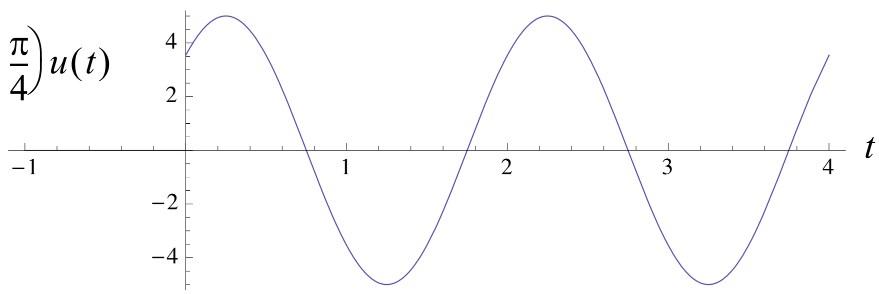
One-Sided Signals

- Another class of signals are those that exist on a semi-infinite interval, i.e., are zero for $t < t_0$ (support $t \in [0, \infty)$)
- The continuous-time unit-step function, $u(t)$, is useful for describing one-sided signals

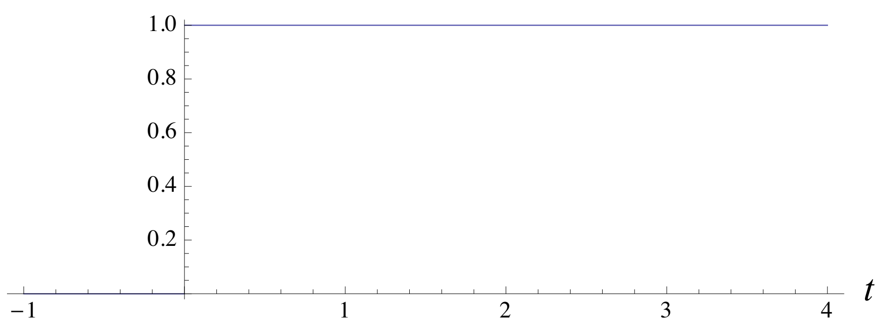
$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{9.3}$$

- When we multiply the previous two-side signals by the step-function a one-side signal is created

$$x(t) = 5\cos\left(2\pi\frac{t}{2} - \frac{\pi}{4}\right)u(t)$$

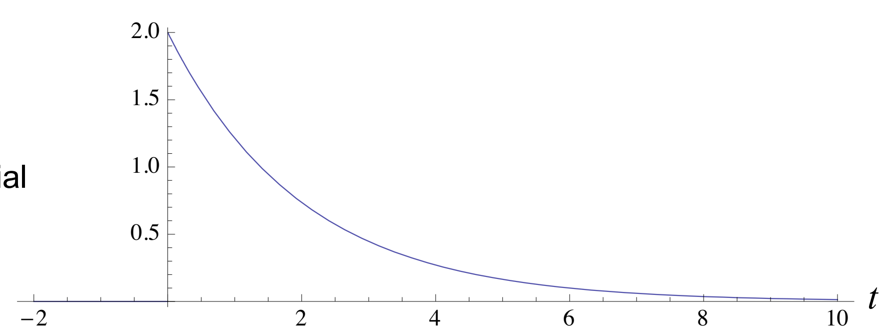


$$x(t) = u(t)$$



$$x(t) = 2e^{-t/2}u(t)$$

One-sided exponential



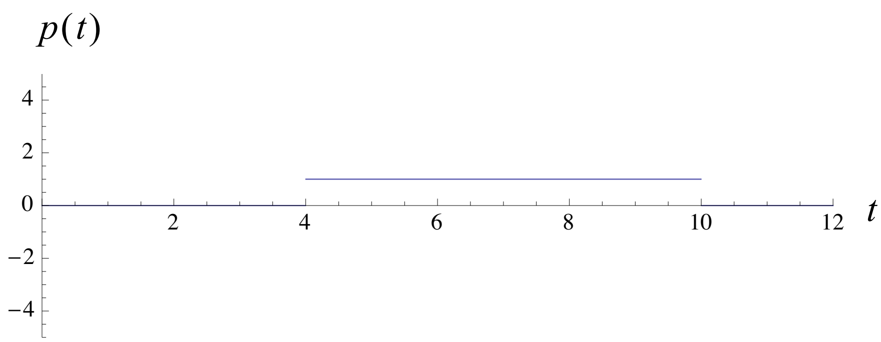
- The start time can easily be changed by letting $t \to t - t_0$

$$x(t) = u(t-2) = \begin{cases} 1, & t \geq 2 \\ 0, & \text{otherwise} \end{cases} \tag{9.4}$$

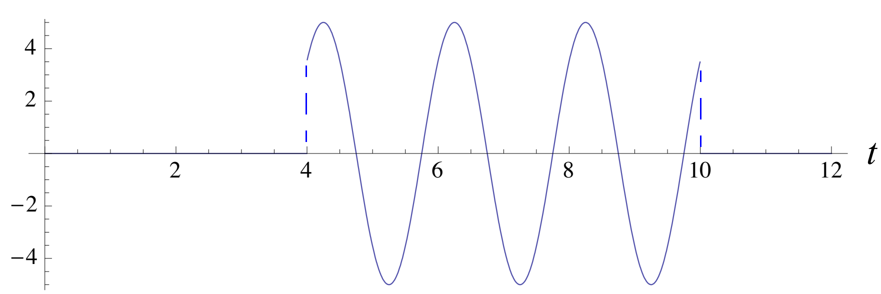
Finite-Duration Signals

- Finite duration signals will have support over just a finite time interval, e.g., $t \in [4, 10)$
- A convenient way of crating such signals is via pulse gating function such as

$$p(t) = u(t-4) - u(t-10) = \begin{cases} 1, & 4 \leq t < 10 \\ 0, & \text{otherwise} \end{cases} \tag{9.5}$$



$$x(t) = 5\cos\left(2\pi\frac{t}{2} - \frac{\pi}{4}\right)p(t)$$



The Unit Impulse

- The topics discussed up to this point have all followed logically from our previous study of discrete-time signals and systems
- The unit impulse signal, $\delta(t)$, however is more difficult to define than the unit impulse sequence, $\delta[n]$
- Recall that

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

- The unit impulse signal is defined as

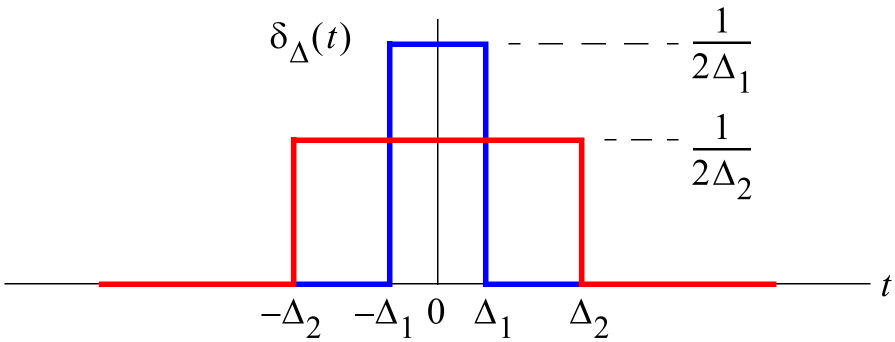
$$\delta(t) = 0, t \neq 0 \tag{9.6}$$

and

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \tag{9.7}$$

- What does this mean?
  - It would seem that $\delta(t)$ must have zero width, yet have area of unity
- A test function, $\delta_\Delta(t)$, can be defined that in fact becomes $\delta(t)$ as $\Delta \to 0$

$$\delta_\Delta(t) = \begin{cases} \frac{1}{2\Delta}, & -\Delta < t < \Delta \\ 0, & \text{otherwise} \end{cases} \tag{9.8}$$



- The claim is that

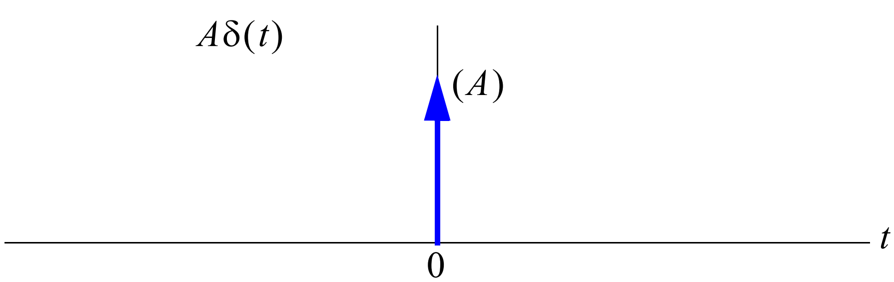
$$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t) \tag{9.9}$$

- Check (9.6) and (9.7)

$$\lim_{\Delta \to 0} \delta_\Delta(t) = 0,\ t \neq 0$$

$$\int_{-\infty}^{\infty} \delta_\Delta(t)dt = 1$$

- In plotting a scaled unit-impulse signal, e.g., $A\delta(t)$, we plot a vertical arrow with the amplitude actually corresponding to the area



Sampling Property of the Impulse

- A noteworthy property of $\delta(t)$ is that

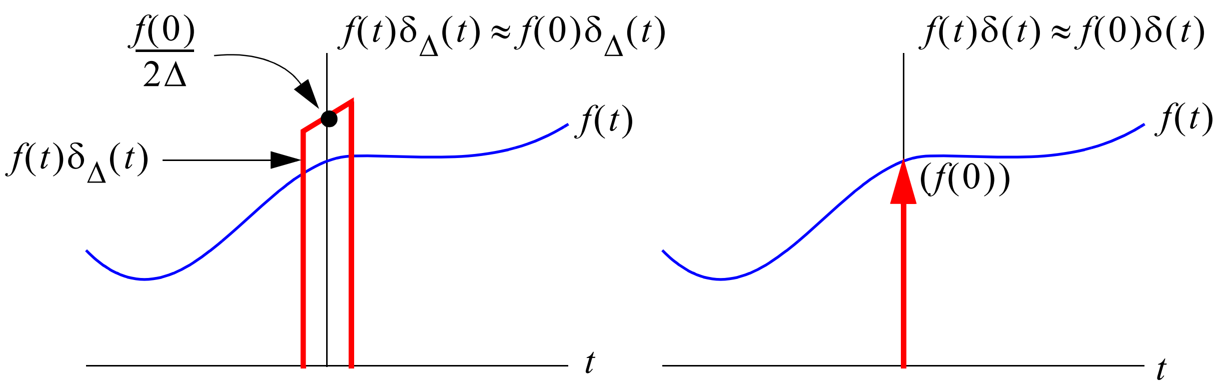
Sampling Property

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0) \tag{9.10}$$

- **Discussion**
  - Since $\delta(t-t_0)$ is zero everywhere except $t = t_0$, only the value $f(t_0)$ is of interest
  - Using the test function $\delta_\Delta(t)$ we also note that

$$f(t)\delta_\Delta(t) = \begin{cases} f(t)/(2\Delta), & -\Delta < t < \Delta \\ 0, & \text{otherwise} \end{cases} \tag{9.11}$$

so as $\Delta \to 0$ the only value of $f(t)$ that matters is $f(0)$



  - Also observe that

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\delta(t)dt &= \int_{-\infty}^{\infty} f(0)\delta(t)dt \\ &= f(0)\int_{-\infty}^{\infty} \delta(t)dt = f(0) \end{aligned} \tag{9.12}$$

- **Integral Form**

Sampling/Sifting Property

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0) \qquad (9.13)$$

<u>Example</u>: $\cos(2\pi t)\delta(t - 1.2) + u(t)\delta(t - 3)$

- The sampling property of $\delta(t)$ results in

$$\cos(2\pi(1.2))\delta(t - 1.2) + u(3)\delta(t - 3)$$

- When integrated we have

$$\int_{-\infty}^{\infty} [\cos(2\pi t)\delta(t - 1.2) + u(t)\delta(t - 3)]dt$$
$$= \cos(2.4\pi) + u(3) = \cos(2.4\pi) + 1$$

Operational Mathematics and the Delta Function

- The impulse function <u>is not</u> a function in the ordinary sense
- It is the most practical when it appears inside of an integral
- From an engineering perspective a true impulse signal <u>does not</u> exist
  - We can create a pulse similar to the test function $\delta_\Delta(t)$ as well as other test functions which behave like impulse functions in the limit
- The operational properties of the impulse function <u>are</u> very useful in continuous-time signals and systems modeling, as well as in probability and random variables, and in modeling distributions in electromagnetics

Derivative of the Unit Step

- A case in point where the operational properties are very valuable is when we consider the derivative of the unit step function
- From calculus you would say that the derivative of the unit step function, $u(t)$, does not exist because of the discontinuity at $t = 0$
- Consider

$$\int_{-\infty}^{t} \delta(\tau)d\tau \tag{9.14}$$

- The *area property* of $\delta(t)$ states that

$$\int_{a}^{b} \delta(t)dt = \begin{cases} 1, & a < 0 \text{ and } b \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{9.15}$$

- Invoking the area property we have

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{9.16}$$

  which says that this integral behaves like the unit step function

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{9.17}$$

- From calculus we recognize that (9.17) implies also that

$$\boxed{\delta(t) = \frac{d}{dt}u(t)} \tag{9.18}$$

- Similarly,

$$\delta(t-t_0) = \frac{d}{dt}u(t-t_0) \tag{9.19}$$

- If we now consider situations where a product exits, i.e., $x(t) = f(t)u(t)$, we can invoke the product rule for derivatives to obtain

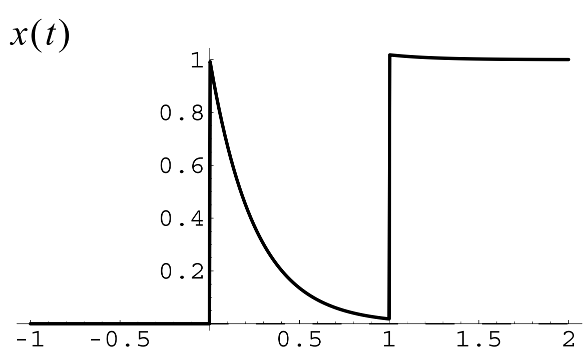
$$\begin{aligned}\frac{d}{dt}f(t)u(t) &= \left(\frac{d}{dt}f(t)\right)u(t) + f(t)\left(\frac{d}{dt}u(t)\right) \\ &= f'(t)u(t) + f(t)\delta(t)\end{aligned} \tag{9.20}$$

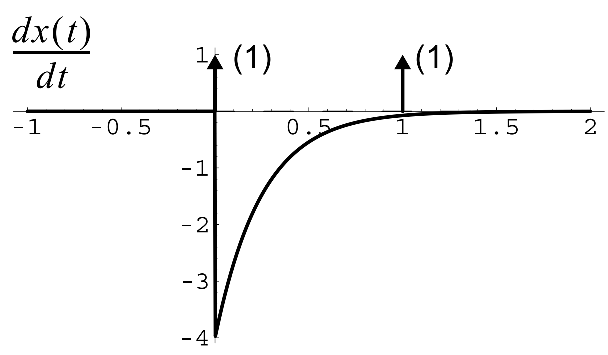
Example: $x(t) = e^{-4t}u(t) + u(t-1)$

- The derivative of $x(t)$ is

$$x'(t) = \frac{d}{dt}x(t) = -4e^{-4t}u(t) + e^{-4t}\delta(t) + \delta(t-1)$$

$$= -4e^{-4t}u(t) + \delta(t) + \delta(t-1)$$

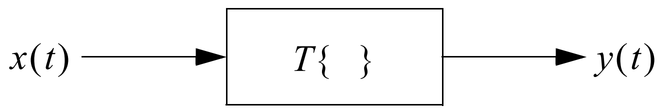




Continuous-Time Systems

- A continuous-time system operates on the input to produce an output

$$y(t) = T\{x(t)\} \tag{9.21}$$



Basic System Examples

Squarer
$$y(t) = [x(t)]^2 \tag{9.22}$$

Time Delay
$$y(t) = x(t - t_d) \tag{9.23}$$

Differentiator
$$y(t) = \frac{dx(t)}{dt} \tag{9.24}$$

Integrator
$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \tag{9.25}$$

- In all of the above we can calculate the output given the input and the definition of the system operator
- For linear time-invariant systems we are particularly interested in the *impulse response*, that is the output, $y(t) = h(t)$, when $x(t) = \delta(t)$, for the system initially at rest

Example: Integrator Impulse Response

- Using the definition

$$y(t) = h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

Linear Time-Invariant Systems

- In the study of discrete-time systems we learned the importance of systems that are linear and time-invariant, and how to verify these properties for a given system operator

Time-Invariance

- A time invariant system obeys the following

$$x(t - t_0) \rightarrow y(t - t_0) \tag{9.26}$$

  for any t_0

- Both the squarer and integrator are time invariant
- The system

$$y(t) = \cos(\omega_c t) x(t) \tag{9.27}$$

  is not time invariant as the gain changes as a function of time

Linearity

- A linear system obeys the following

$$\alpha x_1(t) + \beta x_2(t) \to \alpha y_1(t) + \beta y_2(t) \tag{9.28}$$

  where the inputs are applied together or applied individually and combined via α and β later

- The <u>squarer is nonlinear</u> by virtue of the fact that

$$\begin{aligned} y(t) &= [\alpha x_1(t) + \beta x_2(t)]^2 \\ &= \alpha^2 x_1^2(t) + 2\alpha\beta x_1(t)x_2(t) + \beta^2 x_2(t) \end{aligned}$$

  produces a cross term which does not exist when the two inputs are processed separately and then combined

- The <u>integrator is linear</u> since

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} [\alpha x_1(\tau) + \beta x_2(\tau)]d\tau \\ &= \alpha\int_{-\infty}^{t} x_1(\tau)d\tau + \beta\int_{-\infty}^{t} x_2(\tau)d\tau \end{aligned}$$

The Convolution Integral

- For linear time-invariant (LTI) systems the convolution integral can be used to obtain the output from the input and the system impulse response

Convolution Integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t) \tag{9.29}$$

- The notation used to denote convolution is the same as that used for discrete-time signals and systems, i.e., the convolution sum
- Evaluation of the convolution integral itself can prove to be very challenging

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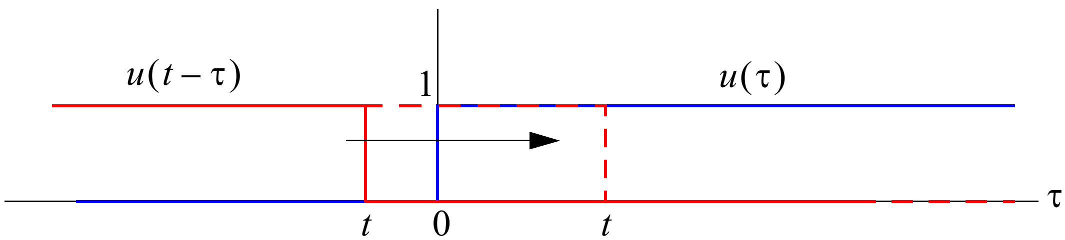
<u>Example</u>: $y(t) = x(t)*h(t) = u(t)*u(t)$

- Setting up the convolution integral we have

$$y(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau$$



$$y(t) = \begin{cases} 0, & t < 0 \\ \int_0^t d\tau, & t \geq 0 \end{cases}$$

$$= \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$

or simply

$$y(t) = tu(t) \equiv r(t),$$

which is known as the *unit ramp*

Properties of Convolution

- Commutativity:

$$x(t)*h(t) = h(t)*x(t) \tag{9.30}$$

- Associativity:

$$[x(t)*h_1(t)]*h_2(t) = x(t)*[h_1(t)*h_2(t)] \tag{9.31}$$

- Distributivity over Addition:

$$x(t)*[h_1(t)*h_2(t)] = x(t)*h_1(t) + x(t)*h_2(t) \tag{9.32}$$

- Identity Element of Convolution:

$$x(t)*h(t) = h(t) \tag{9.33}$$

  What is $x(t)$?

  – It turns out that $x(t) = \delta(t) \Rightarrow \delta(t)*h(t) = h(t)$

  proof

$$\int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} \delta(\tau)h(t-0)d\tau$$

$$= h(t)\int_{-\infty}^{\infty} \delta(\tau)d\tau = h(t)$$

Impulse Response of Basic LTI Systems

- For certain simple systems the impulse response can be found by driving the input with $\delta(t)$ and observing the output
- For complex systems transform techniques, such as the *Laplace transform*, are more appropriate

Integrator

$$h(t) = \int_{-\infty}^{t} x(\tau)d\tau \bigg|_{x(\tau) = \delta(\tau)} = u(t) \tag{9.34}$$

Ideal delay

$$h(t) = x(t - t_d)\Big|_{x(t) = \delta(t)} = \delta(t - t_d) \tag{9.35}$$

- Note that this means that

$$x(t) * \delta(t - t_d) = x(t - t_d) \tag{9.36}$$

Convolution of Impulses

- Basic Theorem:

$$\delta(t - t_1) * \delta(t - t_2) = \delta(t - (t_1 + t_2)) \tag{9.37}$$

Example: $[\delta(t) - 2\delta(t - 3)] * u(t)$

- Using the time shift property (9.36)

$$\delta(t) * u(t) - 2\delta(t - 3) * u(t) = u(t) - 2u(t - 3)$$

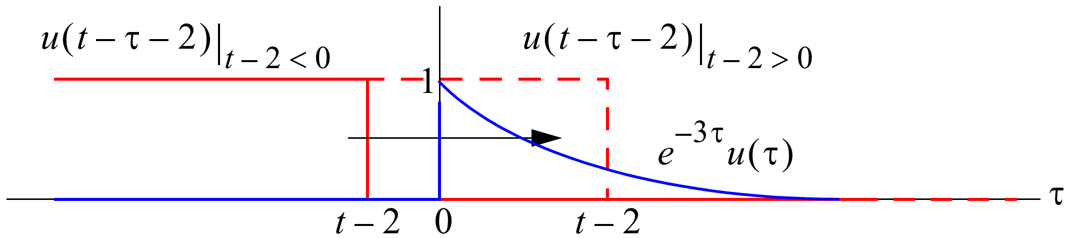
Evaluating Convolution Integrals

Step and Exponential

- Consider $x(t) = u(t - 2)$ and $h(t) = e^{-3t}u(t)$
- We wish to find $y(t) = x(t) * h(t)$

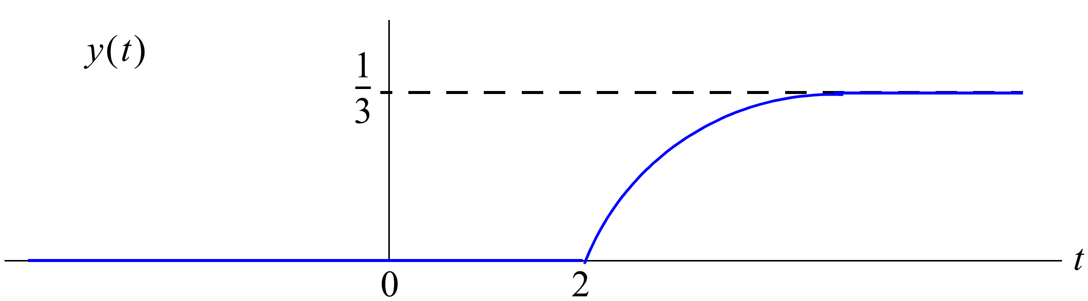
$$y(t) = \int_{-\infty}^{\infty} e^{-3\tau} u(\tau) u(t - \tau - 2) d\tau \tag{9.38}$$

- To evaluate this integral we first need to consider how the step functions in the integrand control the limits of integration

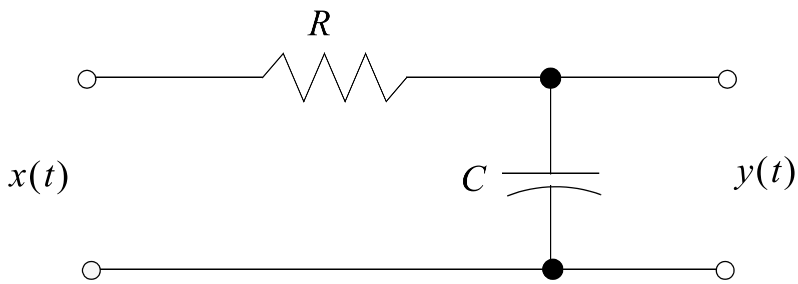


- For $t - 2 < 0$ or $t < 2$ there is no overlap in the product that comprises the integrand, so $y(t) = 0$
- For $t - 2 > 0$ or $t > 2$ there is overlap for $\tau \in [0, t - 2)$, so here

$$\begin{aligned} y(t) &= \int_0^{t-2} e^{-3\tau} d\tau \\ &= \left.\frac{e^{-3\tau}}{-3}\right|_0^{t-2} \\ &= \frac{1}{3}[1 - e^{-3(t-2)}]u(t-2) \end{aligned} \tag{9.39}$$



- **Note**: The use of the exponential impulse response in examples is significant because it occurs frequently in practice, e.g., an *RC* lowpass filter circuit



$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

<u>Example</u>: $x(t) = e^{-at}u(t)$ and $h(t) = e^{-bt}u(t)$

- Find $y(t) = x(t)*h(t)$ by evaluating the convolution integral

$$y(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau$$

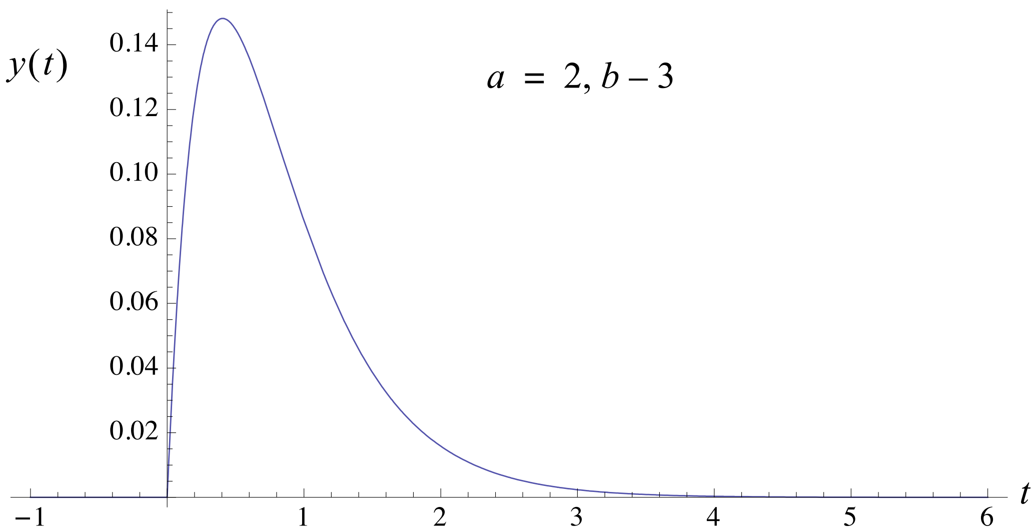
$$= \int_{0}^{t} e^{-a\tau}e^{-b(t-\tau)}d\tau$$

$$= e^{-bt}\int_{0}^{t} e^{-(a-b)\tau}d\tau$$

$$= \frac{e^{-bt}}{a-b}\cdot\frac{e^{-(a-b)\tau}}{-(a-b)}\Bigg|_0^t = \frac{e^{-bt}}{a-b}[1-e^{-(a-b)t}]u(t)$$

$$= \frac{1}{a-b}[e^{-bt}-e^{-at}]u(t),\ a \neq b$$

- Suppose that $a = 2$ and $b = 3$

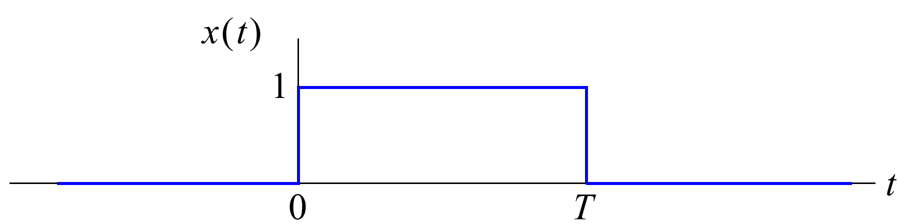


Square-Pulse Input

- Consider a pulse input of the form

$$x(t) = u(t) - u(t-T) \tag{9.40}$$

where T is the pulse width and $h(t) = e^{-at}u(t)$



- The output is

$$y(t) = u(t)*h(t) - u(t-T)*h(t) \tag{9.41}$$

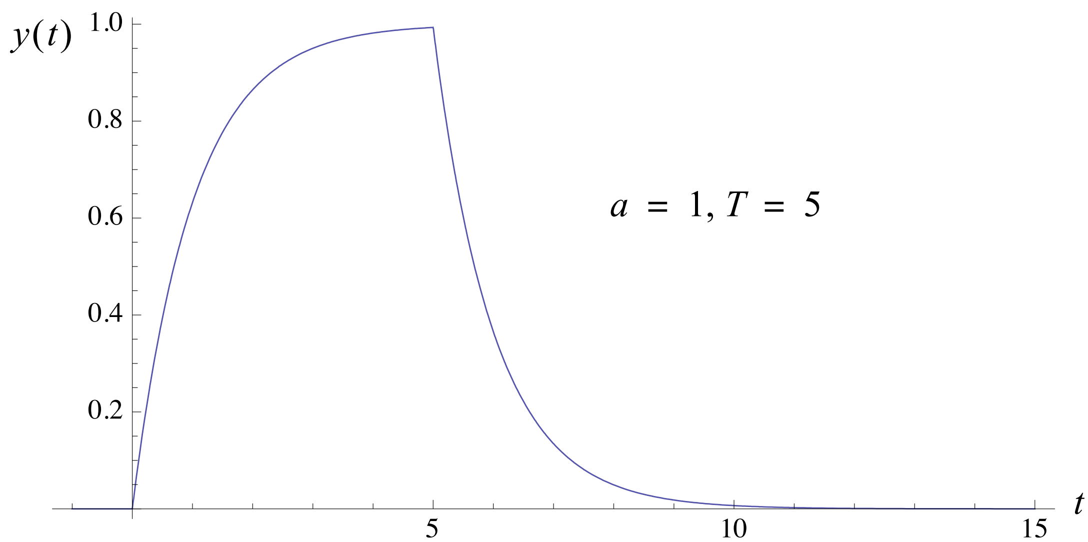
- From the step response analysis we know that

$$u(t)*h(t) = \frac{1}{a}[1-e^{-at}]u(t), \tag{9.42}$$

so

$$y(t) = \frac{1}{a}[1 - a^{-at}]u(t) - \frac{1}{a}[1 - a^{-a(t-T)}]u(t-T) \quad (9.43)$$

- Plot the results for $T = 5$ and $a = 1$

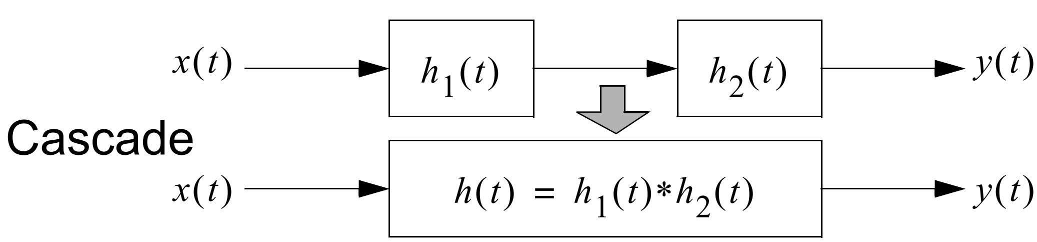


Properties of LTI Systems

Cascade and Parallel Connections

- We have studied cascade and parallel system earlier
- For a cascade of two LTI systems having impulse responses $h_1(t)$ and $h_2(t)$ respectively, the impulse response of the cascade is the convolution of the impulse responses

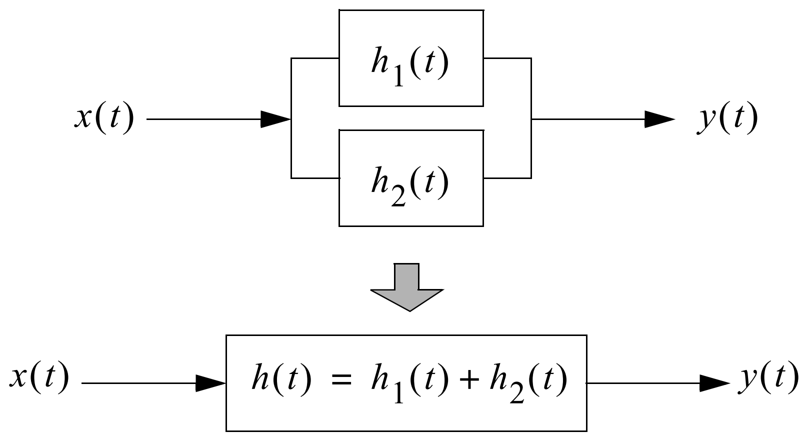
$$h_{\text{cascade}}(t) = h_1(t) * h_2(t) \quad (9.44)$$



- For two systems connected in parallel, the impulse response is the sum of the impulse responses

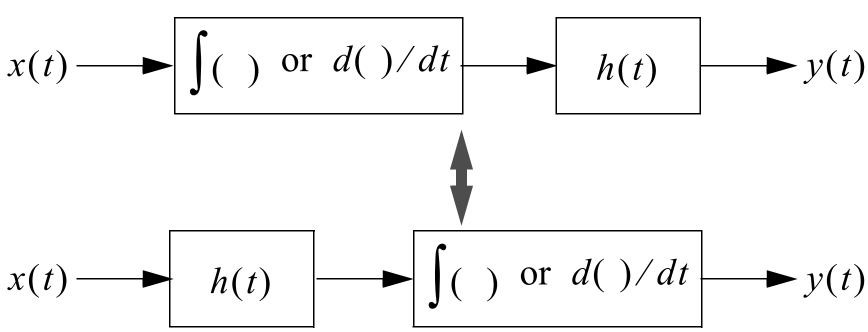
$$h_{\text{parallel}}(t) = h_1(t) + h_2(t) \tag{9.45}$$

Parallel



Differentiation and Integration of Convolution

- Since the integrator and differentiator are both LTI system operations, when used in combination with another system having impulse response, $h(t)$, we find that the cascade property holds
- What this means is that performing differentiation or integration before a signal enters and LTI system, gives the same result as performing the differentiation or integration after the signal passes through the system



Example: Step Response from $h(t) = e^{-at}u(t)$

- Knowing the impulse response of a system we can find the respond to a step input by just integrating the output, since $u(n)$ at the input is obtained by integrating $\delta(t)$
- Thus we can write that

$$y(t) = u(t)*h(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$= \int_{-\infty}^{t} e^{-a\tau}u(\tau)d\tau = \int_{0}^{t} e^{-a\tau}d\tau$$

$$= \left.\frac{e^{-a\tau}}{-a}\right|_{0}^{t} = \frac{1}{a}[1-e^{-at}]u(t)$$

- This result is consistent with earlier analysis

Stability and Causality

- Definition: A system is stable if and only if every bounded input produces a bounded output. A bounded input/output is a signal for which $|x(t)|$ or $|y(t)| < \infty$ for all values of t.
- A theorem which applies to LTI systems states that a system (LTI system) is stable if and only of

Stability for LTI Systems

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty \qquad (9.46)$$

  – *if and only if* holds in either direction

Example: LTI with $h(t) = e^{-at}u(t)$

- For stability

$$\int_{-\infty}^{\infty} \left|e^{-at}u(t)\right| dt = \int_{0}^{\infty} e^{-at} dt$$

$$= \left.\frac{e^{-at}}{-a}\right|_{0}^{\infty} = \frac{1}{a},\ a > 0$$

- We must have $a > 0$ for stability
- Note that $a = 0$ result in $h(t) = u(n)$, which is an integrator system, hence an integrator system is not stable

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- Definition: A system is causal if and only if the output at the present time does not depend upon future values of the input
- A theorem which applies to LTI systems is

Causal for LTI Systems

$$h(t) = 0 \quad \text{for } t < 0 \tag{9.47}$$

- This definition and LTI theorem also holds for discrete-time systems

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Example: Simulate an LTI System using Matlab `lsim()`

- As a final example we consider how we can use MATLAB to simulate LTI systems
- The function we use is `lsim()`, which has behavior similar to that of `filter()`, which is used for discrete-time systems

```
>> t = -1:0.01:15; % create a time axis
>> x = zeros(size(t)); % next 3 lines create a pulse
>> i_pulse = find(t>=0 & t<=5); % duration is 5s
>> x(i_pulse) = ones(size(i_pulse));
>> subplot(211)
>> plot(t,x)
>> axis([-1 15 0 1.1]); grid
>> ylabel('Input x(t)')
>> subplot(212)
>> y = lsim(tf([1],[1 1]),x,t);% h(t) = e^(-1*t) u(t)
Warning: Simulation will start at the nonzero initial
time T(1).
> In lti.lsim at 100
>> plot(t,y); grid
>> ylabel('Output y(t)')
>> xlabel('Time (s)')
```