Frequency Response of FIR Filters

Chapter 6

This chapter continues the study of FIR filters from Chapter 5, but the emphasis is *frequency response*, which relates to how the filter responds to an input of the form

$$x[n] = e^{j\hat{\omega}_0 n}, -\infty < n < \infty .$$

A fundamental result we shall soon see, is that the frequency response and impulse response are related through an operation known as the Fourier transform. The Fourier transform itself is not however formally studied in this chapter.

Sinusoidal Response of FIR Systems

- Consider an FIR filter when the input is a complex sinusoid of the form

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}_0 n}, -\infty < n < \infty, \tag{6.1}$$

where it could be that $x[n]$ was obtained by sampling the complex sinusoid $x(t) = Ae^{j\phi}e^{j\omega_0 t}$ and $\hat{\omega}_0 = \omega_0 T_s$

- From the difference equation for an $M+1$ tap FIR filter,

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k A e^{j\phi} e^{j\hat{\omega}_0(n-k)} \tag{6.2}$$

$$\begin{aligned} y[n] &= Ae^{j\phi}e^{j\hat{\omega}_0 n}\left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}_0 k}\right) \\ &= Ae^{j\phi}e^{j\hat{\omega}_0 n}H(e^{j\hat{\omega}_0}), -\infty < n < \infty \end{aligned} \tag{6.3}$$

where we have defined for arbitrary $\hat{\omega}$

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}} \tag{6.4}$$

to be the *frequency response* of the FIR filter

- Note: The notation $H(e^{j\hat{\omega}})$ is used rather than say $H(\hat{\omega})$, to be consistent with the z-transform which will defined in Chapter 7, and to emphasize the fact that the frequency response is periodic, with period 2π (more on this later)
  - <u>Note</u>: $e^{j\hat{\omega}} = \cos(\hat{\omega}) + j\sin(\hat{\omega})$, where sine and cosine are both $\mathrm{mod}(2\pi)$ functions
- Returning to (6.2), the implication is that when the input is a complex exponential at frequency $\hat{\omega}_0$, the output is also a complex exponential at frequency $\hat{\omega}_0$
- The complex amplitude (magnitude and phase) of the input is changed as a result of passing through the system
  - The frequency response at $\hat{\omega}_0$ multiplies the input amplitude to produce the output
  - It is **not true** in general that $y[n] = H(e^{j\hat{\omega}_0})x[n]$, but only for the special case of $x[n]$ a complex sinusoid applied starting at $-\infty$

- The frequency response is a complex function of $\hat{\omega}$ that is generally viewed in either polar or rectangular form

$$\begin{aligned} H(e^{j\hat{\omega}}) &= \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})} \\ &= \mathrm{Re}\{H(e^{j\hat{\omega}})\} + j\mathrm{Im}\{H(e^{j\hat{\omega}})\} \end{aligned} \tag{6.5}$$

  – The polar or magnitude and phase form is perhaps the most common

- The polar form offers the following interpretation of $y[n]$ in terms of $x[n]$, when the input is a complex sinusoid

$$\begin{aligned} y[n] &= \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})} \cdot A e^{j\phi} e^{j\hat{\omega}n} \\ &= \left|H(e^{j\hat{\omega}})\right| A e^{j[\angle(H(e^{j\hat{\omega}}) + \phi)]} e^{j\hat{\omega}n} \end{aligned} \tag{6.6}$$

  – Here we see that the input amplitude is multiplied by the frequency response amplitude, and the input phase ϕ has added to it the frequency response phase

  – The output amplitude expression means that $\left|H(e^{j\hat{\omega}})\right|$ is also termed the *gain* of an LTI system

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<u>Example</u>: $\{b_k\} = \{1, 1, 3, 1, 1\}$

- The frequency response of this FIR filter is

$$\begin{aligned} H(e^{j\hat{\omega}}) &= \sum_{k=0}^{4} b_k e^{-j\hat{\omega}k} \\ &= 1 + e^{-j\hat{\omega}} + 3e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}} \end{aligned}$$

$$H(e^{j\hat{\omega}}) = e^{-j2\hat{\omega}}[e^{j2\hat{\omega}} + e^{j\hat{\omega}} + 3 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}]$$

$$= e^{-j2\hat{\omega}}[2\cos(2\hat{\omega}) + 2\cos(\hat{\omega}) + 3]$$

– We have used the inverse Euler formula for cosine twice

- For this particular filter we have that

$$\left|H(e^{j\hat{\omega}})\right| = 3 + 2\cos(\hat{\omega}) + 2\cos(2\hat{\omega})$$

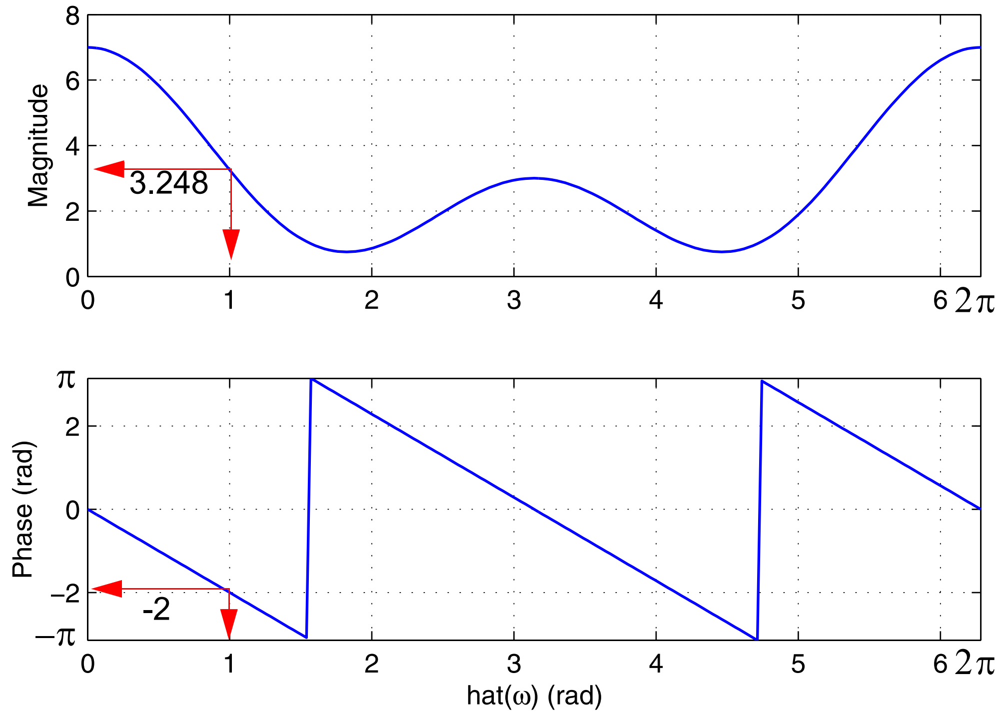
$$\angle H(e^{j\hat{\omega}}) = -2\hat{\omega}$$

Why?

- Use MATLAB to plot the magnitude and phase response

```
>> w = 0:2*pi/200:2*pi;
>> H = exp(-j*2*w).*(3 + 2*cos(w) + 2*cos(2*w));
>> subplot(211)
>> plot(w,abs(H))
>> axis([0 2*pi 0 8])
>> grid
>> ylabel('Magnitude')
>> subplot(212)
>> plot(w,angle(H))
>> axis([0 2*pi -pi pi])
>> grid
>> ylabel('Phase (rad)')
```

```
>> xlabel('hat(\omega) (rad)')
```



Example: Find $y[n]$ for Input $x[n] = 5e^{j(1 \cdot n)}$

- The input frequency is $\hat{\omega}_0 = 1$ rad, the amplitude is 5, and the phase is $\phi = 0$
- Assuming $x[n]$ is input to the 4-tap FIR filter in the previous example, the filter output is

$$y[n] = (3 + 2\cos(1) + 2\cos(2 \cdot 1))e^{-j(1 \cdot 2)}5e^{j(1 \cdot n)}$$
$$= 16.2415e^{-j2} \cdot e^{jn}$$

- The amplitude response or gain at $\hat{\omega}_0 = 1$ is $|H(e^{j1})| = 3.248$; why?

Superposition and the Frequency Response

- We can use the linearity of the FIR filter to compute the output to a sum of sinusoids input signal
- As a special case we first consider a single real sinusoid

$$x[n] = A\cos(\hat{\omega}_0 n + \phi) \tag{6.7}$$

- Using Euler's formula we expand (6.7)

$$x[n] = \frac{A}{2}e^{j(\hat{\omega}_0 n + \phi)} + \frac{A}{2}e^{-j(\hat{\omega}_0 n + \phi)} \tag{6.8}$$

- The filter output due to each complex sinusoid is known from (6.3), so now using superposition we can write

$$y[n] = \frac{A}{2}H(e^{j\hat{\omega}_0})e^{j(\hat{\omega}_0 n + \phi)} + \frac{A}{2}H(e^{-j\hat{\omega}_0})e^{-j(\hat{\omega}_0 n + \phi)} \tag{6.9}$$

- We can simplify this result to a nice compact form, if we make the assumption that the FIR filter has real coefficients
- **Special Result**: It will be shown in a later section of this chapter that an FIR filter with real coefficients has *conjugate symmetry*

$$\boxed{H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})} \tag{6.10}$$

  – What does this mean?

$$\begin{aligned} H(e^{-j\hat{\omega}}) &= \mathrm{Re}\{H(e^{j\hat{\omega}})\} - j\mathrm{Im}\{H(e^{j\hat{\omega}})\} \\ &= \left|H(e^{j\hat{\omega}})\right|\{-\angle H(e^{j\hat{\omega}})\} \end{aligned}$$

- We now use (6.10) to simplify (6.9)

$$y[n] = \frac{A}{2}H(e^{j\hat{\omega}_0})e^{j(\hat{\omega}_0 n + \phi)} + \frac{A}{2}H^*(e^{j\hat{\omega}_0})e^{-j(\hat{\omega}_0 n + \phi)}$$

$$= A\left|H(e^{j\hat{\omega}_0})\right| \cdot \left[\frac{e^{j(\hat{\omega}_0 n + \phi + \angle H(e^{j\hat{\omega}}))} + e^{-j(\hat{\omega}_0 n + \phi + \angle H(e^{j\hat{\omega}}))}}{2}\right] \tag{6.11}$$

$$y[n] = A\left|H(e^{j\hat{\omega}_0})\right|\cos[\hat{\omega}_0 n + \phi + \angle H(e^{j\hat{\omega}_0})]$$

- We see that when a real sinusoid passes through an LTI system, such as an FIR filter (having real coefficients), the output is also a real sinusoid which has picked up the magnitude and phase of the system at $\hat{\omega} = \hat{\omega}_0$
- The generalization (sum of sinusoids) of this result is when

$$x[n] = X_0 + \sum_{k=1}^{N} |X_k|\cos(\hat{\omega}_k n + \angle X_k), \tag{6.12}$$

then the corresponding LTI system output is

$$y[n] = X_0 H(e^{j0}) + \sum_{k=1}^{N} |X_k|\left|H(e^{j\hat{\omega}_k})\right|\cos[\hat{\omega}_k n + \angle X_k + \angle H(e^{j\hat{\omega}_k})] \tag{6.13}$$

Example: Three Inputs with $b_k = \{1, -1, 1\}$

- The input is

$$x[n] = 10 + 4\cos\left(\frac{\pi}{4}n + \frac{\pi}{8}\right) + 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{4}\right)$$

- The frequency response is

$$H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$
$$= e^{-j\hat{\omega}}[-1 + 2\cos(\hat{\omega})]$$

- The input frequencies are $\hat{\omega}_k = \{0, \pi/4, \pi/3\}$

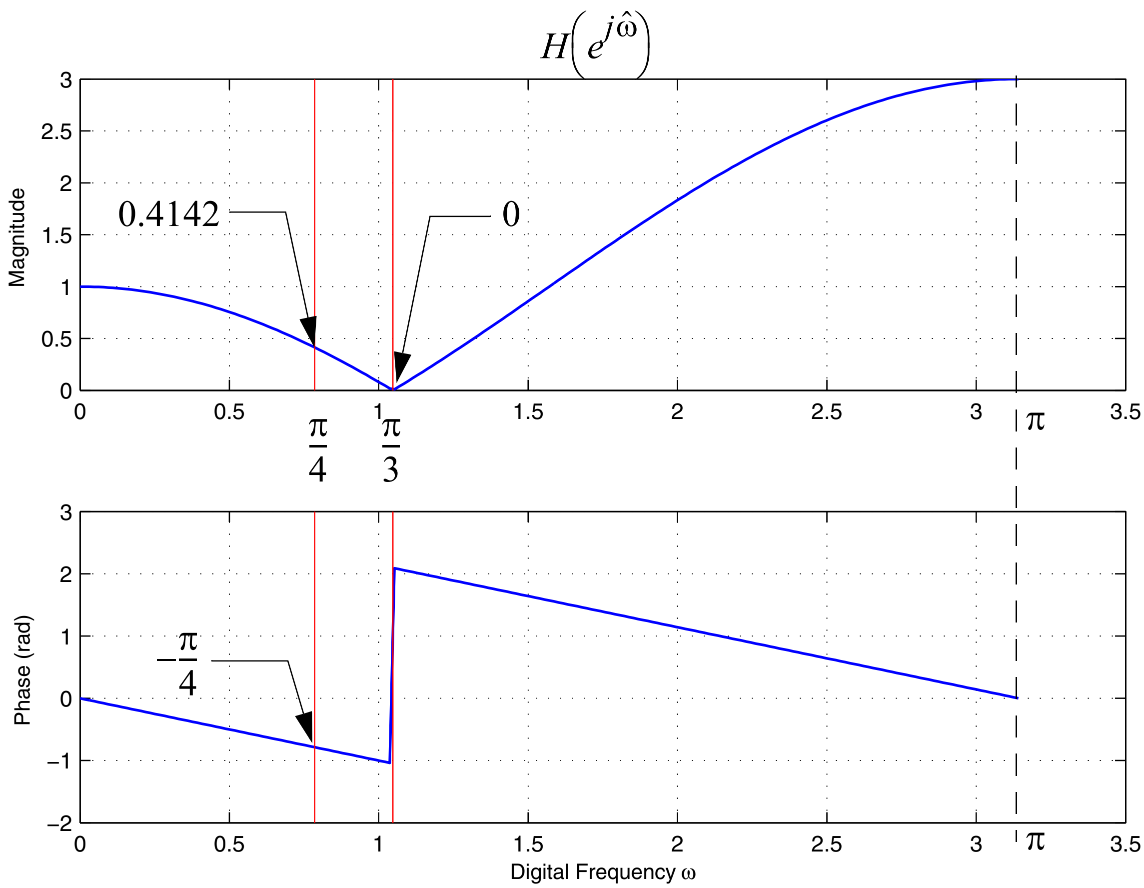
$$H(e^{j0}) = e^{-j0}[-1 + 2\cos(0)] = 1$$
$$H(e^{j\pi/4}) = e^{-j\pi/4}[-1 + 2\cos(\pi/4)] = 0.4142e^{-j\pi/4}$$
$$H(e^{j\pi/3}) = e^{-j\pi/3}[-1 + 2\cos(\pi/3)] = 0$$

- Thus

$$y[n] = 10 \cdot 1 + 4 \cdot 0.4142\cos\left(\frac{\pi}{4}n + \frac{\pi}{8} - \frac{\pi}{4}\right)$$
$$= 10 + 1.6569\cos\left(\frac{\pi}{4}n - \frac{\pi}{8}\right)$$

```
>> w = 0:pi/200:pi;
>> H = 1 - exp(-j*w) + exp(-j*2*w);
>> subplot(211)
>> plot(w,abs(H))
>> grid
>> hold
>> plot([pi/4 pi/4],[0 3],'r')
```

```
>> plot([pi/3 pi/3],[0 3],'r')
>> ylabel('Magnitude')
>> subplot(212)
>> plot(w,angle(H))
>> grid
>> hold
>> plot([pi/4 pi/4],[-2 3],'r')
>> ylabel('Phase (rad)')
>> xlabel('Digital Frequency \omega')
```



- We have used frequency domain analysis to complete this example
- We could also perform a time domain analysis using say MATLAB's filter function (more on this in the next section)

Steady-State and Transient Response

- The frequency response definition relies on the fact that the support interval for the complex sinusoid input is $[-\infty, \infty]$
- In a computer simulation or in a real-time signal processing application, it is not practical to consider input signals which begin at $-\infty$
- Consider an input that begins at $n = 0$ using the unit step function to *turn on* the input

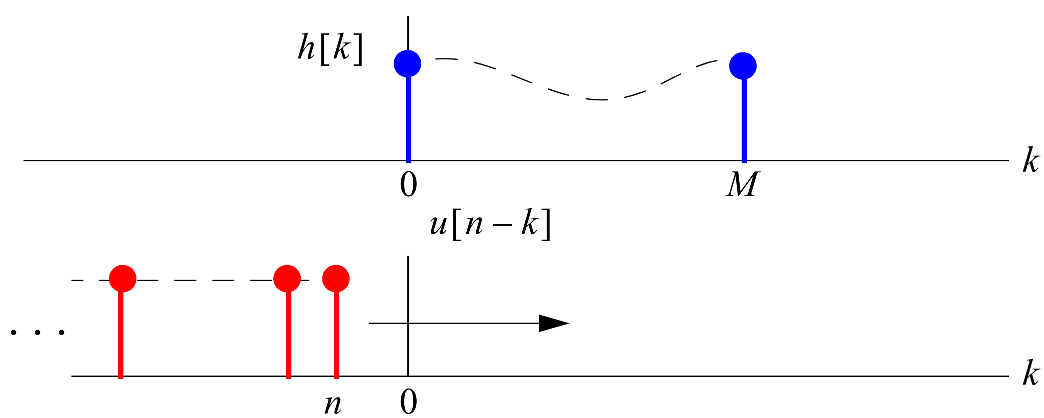
$$x[n] = Xe^{j\hat{\omega}n}u[n] \tag{6.14}$$

where $X = Ae^{j\phi}$ and

$$u[n] \equiv \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{6.15}$$

- For an LTI FIR system, the convolution sum formula yields

$$y[n] = \sum_{k=0}^{M} b_k X e^{j\hat{\omega}(n-k)} u[n-k] \tag{6.16}$$



- The sum of (6.16) is composed of three cases

$$y[n] = \begin{cases} 0, & n < 0 \\ \left(\sum_{k=0}^{n} b_k e^{-j\hat{\omega}k}\right) X e^{j\hat{\omega}n}, & 0 \le n < M \\ \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right) X e^{j\hat{\omega}n}, & n \ge M \end{cases} \qquad (6.17)$$

- When $n < 0$ the input is not present, so the output is zero
- When $0 \le n < M$ the input has partially engaged the filter and the output produced is the *transient response*
- When $n \ge M$, the output is in *steady-state*, since the input has fully engaged the filter
  - Note that the steady-state response is also equivalent to saying

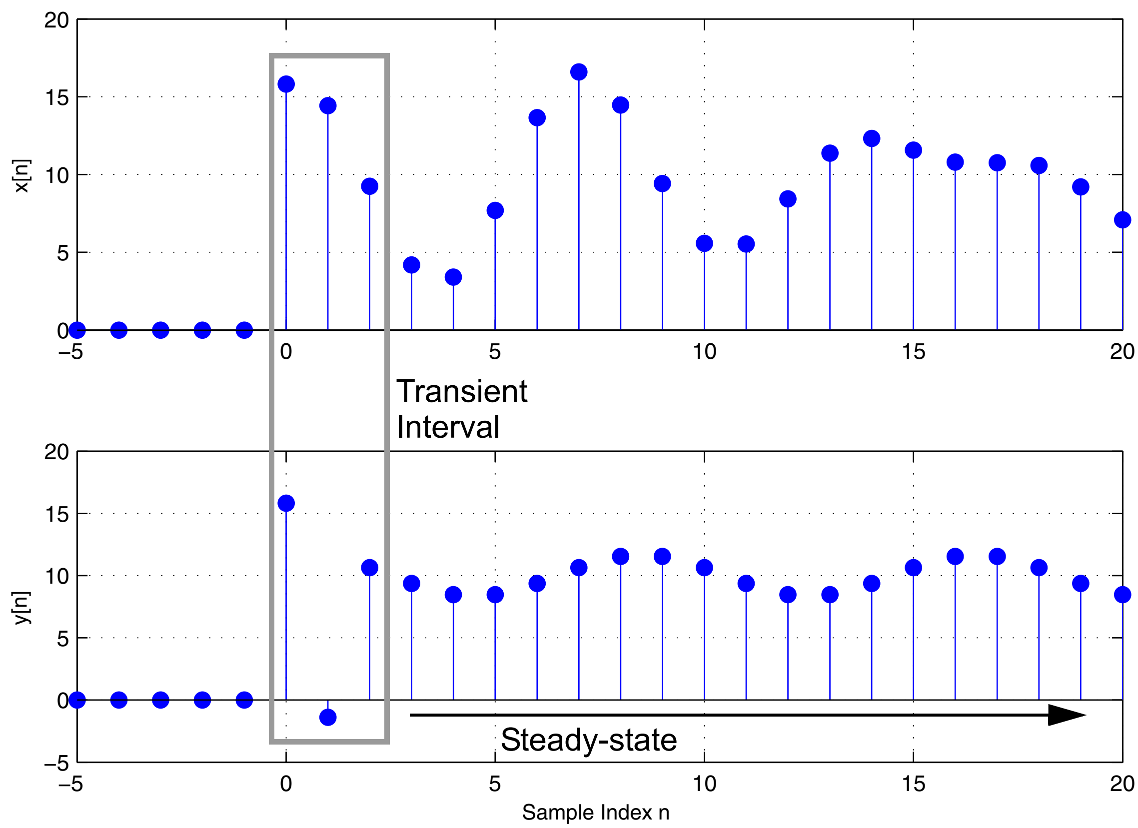
$$y[n] = H(e^{j\hat{\omega}}) X e^{j\hat{\omega}}, n \ge M \qquad (6.18)$$

  - Note also that the steady-state output assumes that the input is always $Xe^{j\hat{\omega}n}$ for $n \ge 0$

Example: Revisit Three Inputs with $b_k = \{1, -1, 1\}$

- The frequency domain analysis used in the previous example obtained just the steady-state portion of the output
- We can evaluate the convolution sum or use MATLAB's filter function to obtain a solution of the form described in (6.17)

```
>> n = -5:20;
>> x = (10 + 4*cos(pi/4*n+pi/8) + ...
     3*cos(pi/3*n-pi/4)).*ustep(n,0);
>> y = filter([1 -1 1],1,x);
>> subplot(211)
>> stem(n,x,'filled')
>> grid
>> ylabel('x[n]')
>> subplot(212)
>> stem(n,y,'filled')
>> hold
Current plot released
>> stem(n,y,'filled')
>> grid
>> ylabel('y[n]')
>> xlabel('Sample Index n')
```

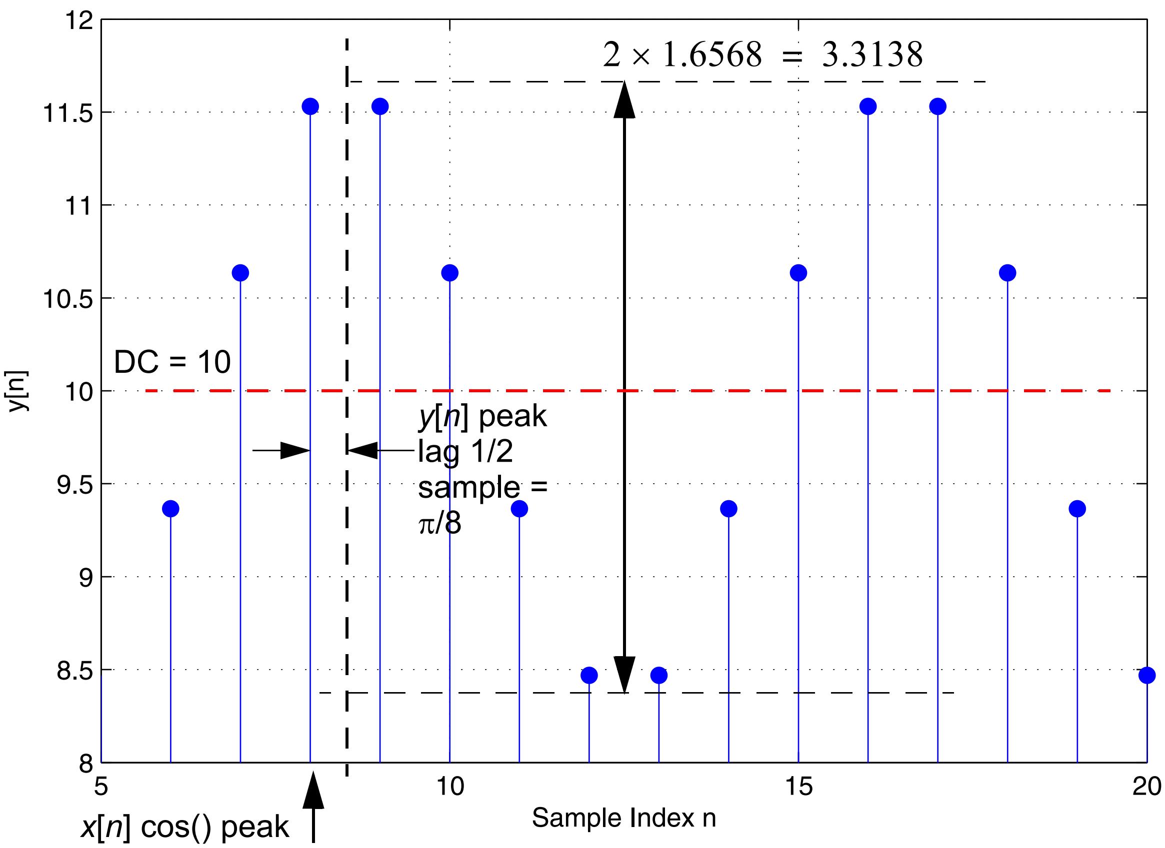


- The function ustep() is defined as:

```
function x = ustep(n,n0)
% function x = ustep(n,n0)
%
% Generate a time shifted unit step sequence

x = zeros(size(n));
i_n0 = find(n >= n0);
x(i_n0) = ones(size(i_n0));
```

- If we consider just the steady-state portion of the output we should be able to discern the same output as obtained from the frequency domain analysis of the previous example



- The measurements above agree with the earlier example

Properties of the Frequency Response

Relation to Impulse Response and Difference Equation

- In the definition of the frequency response, $H(e^{j\hat{\omega}})$, for an FIR filter, the coefficients $\{b_k\}$ were utilized
- Similarly given the frequency response, the impulse response can be found

Time Domain $\leftrightarrow$ Frequency Domain

$$h[n] = \sum_{k=0}^{M} h[k]\delta[n-k] \leftrightarrow H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}$$

- In Chapter 5 we saw that the impulse response and difference equation are directly related, so in summary given one the other two are easy to obtain

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Example: $h[n] \leftrightarrow H(e^{j\hat{\omega}})$

- Given say

$$h[n] = 2\delta[n] + 3\delta[n-1] - 2\delta[n-2]$$

we immediately see that $\{b_k\} = \{2, 3, -2\}$

- The difference equation is

$$y[n] = 2x[n] + 3x[n-1] - 2x[n-2]$$

- The frequency response is

$$H(e^{j\hat{\omega}}) = 2 + 3e^{-j\hat{\omega}} - 2e^{-j2\hat{\omega}}$$

Example: Difference Equation from $H(e^{j\hat{\omega}})$

- Suppose that

$$H(e^{j\hat{\omega}}) = 1 + 4e^{-j2\hat{\omega}} + 4e^{-j4\hat{\omega}}$$

- We immediately can write that

$$h[n] = \delta[n] + 4\delta[n-2] + 4\delta[n-4]$$

and

$$y[n] = x[n] + 4x[n-2] + 4x[n-4]$$

- It could have been that

$$H(e^{j\hat{\omega}}) = 1 + 8e^{-j2\hat{\omega}}\left(\frac{e^{j2\hat{\omega}} + e^{-j2\hat{\omega}}}{2}\right)$$

$$= 1 + 8e^{-j2\hat{\omega}}\cos(2\hat{\omega})$$

started in this form

Example: $H(e^{j\hat{\omega}}) = 2j\sin(\hat{\omega}/2)e^{-j\hat{\omega}/2}$

- We need to convert the sine form back to complex exponentials

$$H(e^{j\hat{\omega}}) = 2je^{-j(\hat{\omega}/2)}\left(\frac{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}}{2j}\right)$$

$$= 1 - e^{-j\hat{\omega}}$$

- The impulse response is thus?

- The difference equation is?

Periodicity of $H(e^{j\hat{\omega}})$

- For any discrete-time LTI system, the frequency response is periodic, that is

$$\boxed{H(e^{j(\hat{\omega}+2\pi)}) \equiv H(e^{j\hat{\omega}})} \tag{6.19}$$

- The proof for FIR filters follows

$$\begin{aligned} H(e^{j(\hat{\omega}+2\pi)}) &= \sum_{k=0}^{M} b_k e^{-j(\hat{\omega}+2\pi)} \\ &= \sum_{k=0}^{M} b_k e^{-j\hat{\omega}} \underbrace{e^{-j2\pi}}_{1} = H(e^{j\hat{\omega}}) \end{aligned} \tag{6.20}$$

- This result is consistent with the fact that sinusoidal signals are insensitive to 2π frequency shifts, i.e.,

$$x[n] = Xe^{j(\hat{\omega}+2\pi)n} = Xe^{j\hat{\omega}n}e^{j2\pi n} = Xe^{j\hat{\omega}n}$$

- In summary, the frequency response is unique on at most a 2π interval, say in particular the interval $-\pi < \hat{\omega} \le \pi$

Conjugate Symmetry

- The frequency response $H(e^{j\hat{\omega}})$ is in general a complex quantity, in most cases obeys certain symmetry properties

- In particular if the coefficients of a digital filter are real, that is $b_k = b_k^*$ or $h[k] = h^*[k]$, then

$$\boxed{\text{Conjugate Symmetry: } H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})} \tag{6.21}$$

<u>proof</u>

$$\begin{aligned} H^*(e^{j\hat{\omega}}) &= \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right)^* = \sum_{k=0}^{M} b_k^* e^{j\hat{\omega}k} \\ &= \sum_{k=0}^{M} b_k e^{-j(-\hat{\omega})k} = H(e^{-j\hat{\omega}}) \end{aligned} \tag{6.22}$$

- A consequence of conjugate symmetry is that

$$\begin{aligned} \left|H(e^{-j\hat{\omega}})\right| &= \left|H(e^{j\hat{\omega}})\right| \\ \angle H(e^{-j\hat{\omega}}) &= -\angle H(e^{j\hat{\omega}}) \end{aligned} \tag{6.23}$$

  – We say that the magnitude response is an even function of $\hat{\omega}$ and the phase is an odd function of $\hat{\omega}$

- It also follows that

$$\begin{aligned} \mathrm{Re}\{H(e^{-j\hat{\omega}})\} &= \mathrm{Re}\{H(e^{j\hat{\omega}})\} \\ \mathrm{Im}\{H(e^{-j\hat{\omega}})\} &= -\mathrm{Im}\{H(e^{j\hat{\omega}})\} \end{aligned} \tag{6.24}$$

  – We say that the real part response is an even function of $\hat{\omega}$ and the imaginary part is an odd function of $\hat{\omega}$

- When plotting the frequency response we can use the above symmetry to just plot over the interval $0 \le \hat{\omega} \le \pi$

Graphical Representation of the Frequency Response

- A useful MATLAB function for plotting the frequency response of any discrete-time (digital) filter is `freqz()`
- The interface to `freqz()` is similar to `filter()` in that `a` and `b` vectors are again required
  - Recall that the `b` vector holds the FIR coefficients $\{b_k\}$ and for FIR filters we set `a = 1`.

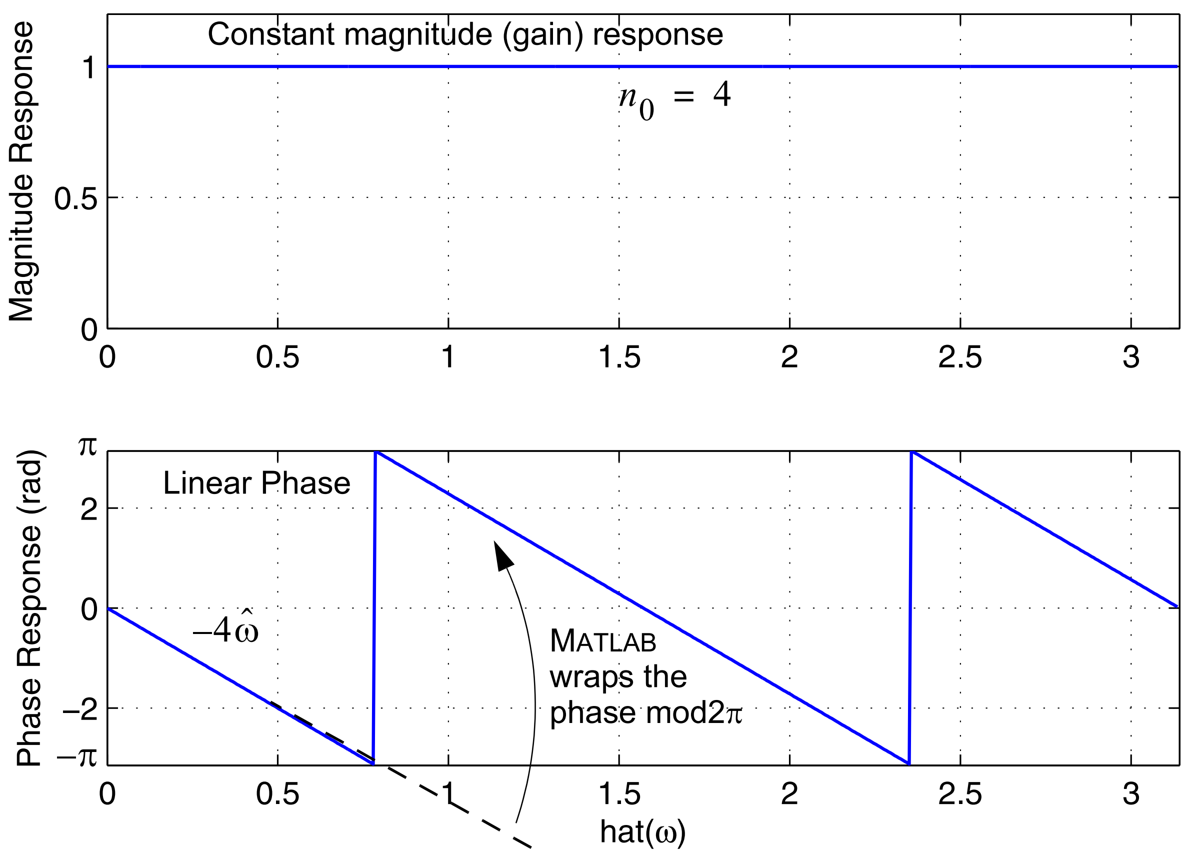
```
>> help freqz
 FREQZ Digital filter frequency response.
    [H,W] = FREQZ(B,A,N) returns the N-point complex frequency response
    vector H and the N-point frequency vector W in radians/sample of
    the filter:
                jw               -jw              -jmw
         jw  B(e)    b(1) + b(2)e + .... + b(m+1)e
      H(e) = ---- = ------------------------------------
                jw               -jw              -jnw
             A(e)    a(1) + a(2)e + .... + a(n+1)e
    given numerator and denominator coefficients in vectors B and A. The
    frequency response is evaluated at N points equally spaced around the
    upper half of the unit circle. If N isn't specified, it defaults to
    512.

    [H,W] = FREQZ(B,A,N,'whole') uses N points around the whole unit circle.

    H = FREQZ(B,A,W) returns the frequency response at frequencies
    designated in vector W, in radians/sample (normally between 0 and pi).

    [H,F] = FREQZ(B,A,N,Fs) and [H,F] = FREQZ(B,A,N,'whole',Fs) return
    frequency vector F (in Hz), where Fs is the sampling frequency (in Hz).

    H = FREQZ(B,A,F,Fs) returns the complex frequency response at the
    frequencies designated in vector F (in Hz), where Fs is the sampling
    frequency (in Hz).

    FREQZ(B,A,...) with no output arguments plots the magnitude and
    unwrapped phase of the filter in the current figure window.

    See also filter, fft, invfreqz, fvtool, and freqs.
```

Example: Delay System $y[n] = x[n-n_0]$

- This filter contains one coefficient, $b_{n_0} = 1$, so

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_0} = 1\angle -\hat{\omega}n_0$$

```
>> [H,w] = freqz([0 0 0 0 1],1);
>> subplot(211)
>> plot(w,abs(H))
>> axis([0 pi 0 1.2]); grid
>> ylabel('Magniude Response')
>> subplot(212)
>> plot(w,angle(H))
>> axis([0 pi -pi pi]); grid
>> ylabel('Phase Response (rad)')
>> xlabel('hat(\omega)')
```

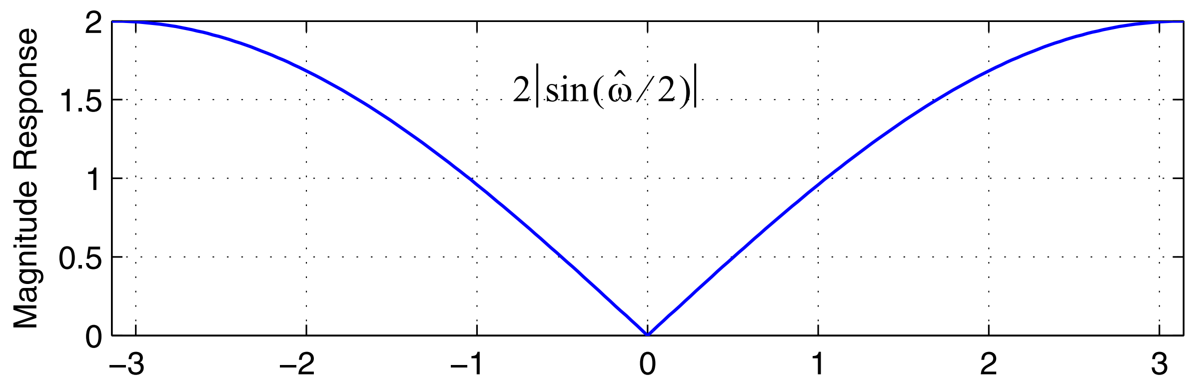


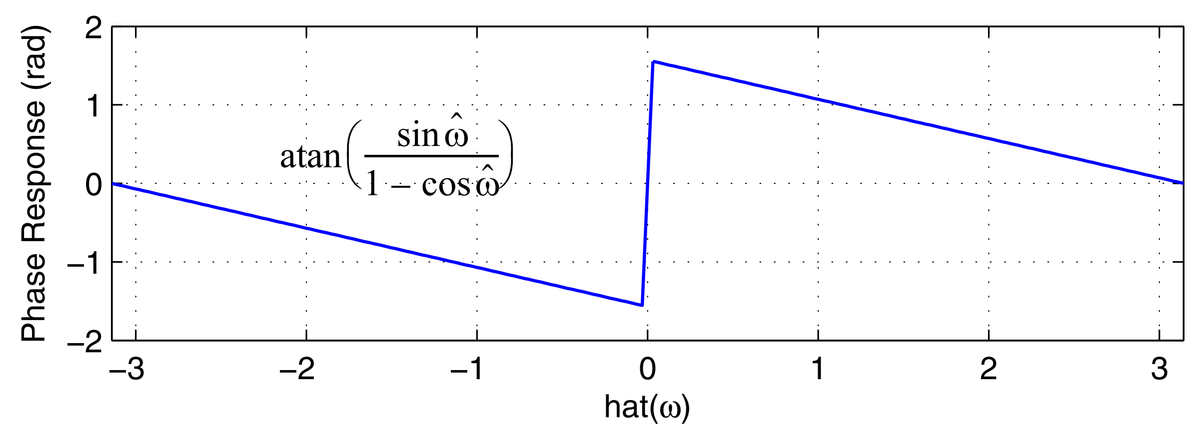
Example: First-Difference System $y[n] = x[n] - x[n-1]$

- The frequency response is

$$H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}} = (1 - \cos\hat{\omega}) + j\sin\hat{\omega}$$

```
>> w = -pi:pi/100:pi;
>> H = freqz([1 -1],1,w); %use a custom w axis
>> subplot(211)
>> plot(w,abs(H))
>> axis([-pi pi 0 2]); grid; ylabel('Magniude Response')
>> subplot(212)
>> plot(w,angle(H))
>> axis([-pi pi -2 2]); grid;
>> ylabel('Phase Response (rad)')
>> xlabel('hat(\omega)')
```





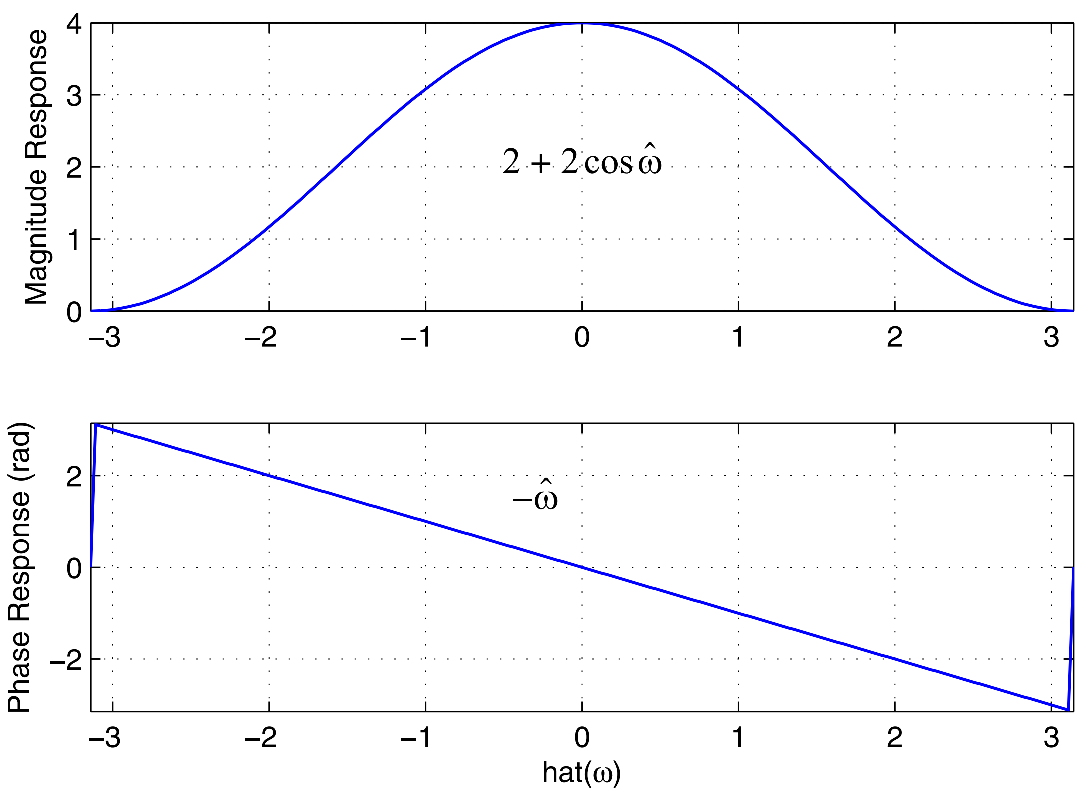
Example: A Simple Lowpass Filter

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$

- The frequency response is

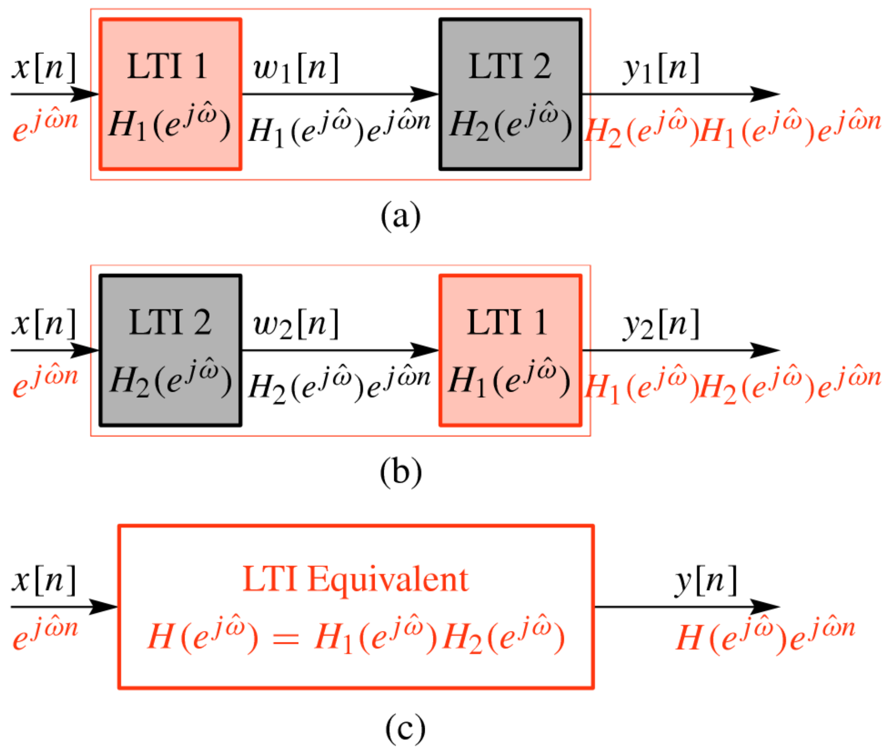
$$H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{j2\hat{\omega}} = e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

```
>> w = -pi:pi/100:pi;
>> H = freqz([1 2 1],1,w);
>> subplot(211)
>> plot(w,abs(H))
>> axis([-pi pi 0 4]); grid;
>> ylabel('Magnitude Response')
>> subplot(212)
>> plot(w,angle(H))
>> axis([-pi pi -pi pi]); grid;
>> ylabel('Phase Response (rad)')
>> xlabel('hat(\omega)')
```



Cascaded LTI Systems

- Cascaded LTI system were discussed in Chapter 5 from the time-domain standpoint
- In the frequency-domain there is an alternative view of the input/output relationships



- To begin with consider again

$$x[n] = e^{j\hat{\omega}n}, -\infty < n < \infty \tag{6.25}$$

- With LTI #1 followed by LTI #2, the output at frequency $\hat{\omega}$ is

$$\begin{aligned} y_1[n] &= H_2(e^{j\hat{\omega}})(H_1(e^{j\hat{\omega}})e^{j\hat{\omega}n}) \\ &= H_2(e^{j\hat{\omega}})H_1(e^{j\hat{\omega}})e^{j\hat{\omega}n} \end{aligned} \tag{6.26}$$

- With LTI #2 followed by LTI #1, the output at frequency $\hat{\omega}$ is

$$\begin{aligned} y_2[n] &= H_1(e^{j\hat{\omega}})(H_2(e^{j\hat{\omega}})e^{j\hat{\omega}n}) \\ &= H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})e^{j\hat{\omega}n} \end{aligned} \tag{6.27}$$

- Since $H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}}) = H_2(e^{j\hat{\omega}})H_1(e^{j\hat{\omega}})$, it follows that $y_1[n] = y_2[n]$, and we have established that the frequency response for a cascaded system is simply the product of the frequency responses

$$\boxed{H_{\text{cascade}}(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})} \tag{6.28}$$

- We have also shown that a convolution in the time-domain is equivalent to a multiplication in the frequency-domain

$$\begin{gathered} \text{Time Domain} \leftrightarrow \text{Frequency Domain} \\ h[n] = h_1[n] * h_2[n] \leftrightarrow H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}}) \end{gathered} \tag{6.29}$$

<u>Example</u>: Two System Cascade

- Consider

$$\begin{aligned} H_1(e^{j\hat{\omega}}) &= 1 + e^{-j\hat{\omega}} \\ H_2(e^{j\hat{\omega}}) &= 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \end{aligned}$$

- The cascade frequency response is

$$\begin{aligned} H(e^{j\hat{\omega}}) &= (1 + e^{-j\hat{\omega}})(1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}) \\ &= 1 + 3e^{-j\hat{\omega}} + 3e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} \end{aligned}$$

- Given the cascade frequency response we can easily convert back to the impulse response of the cascade difference equation

$$h[n] = \delta[n] + 3\delta[n-1] + 3\delta[n-2] + \delta[n-3]$$

and

$$y[n] = x[n] + 3x[n-1] + 3x[n-2] + x[n-3]$$

- Check $h_1[n] * h_2[n]$

Moving Average Filtering

- The moving average filter occurs frequency enough that we should consider finding a general expression for the frequency response
- The difference equation for an L-point averager is

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \tag{6.30}$$

- The frequency response is

$$H(e^{j\hat{\omega}}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k} \tag{6.31}$$

- This sum is known as a *finite geometric series*
- It can be shown (discussed in class) that

$$\sum_{k=0}^{L-1} \alpha^k = \begin{cases} \dfrac{1-\alpha^L}{1-\alpha}, & \alpha \neq 1 \\ L, & \alpha = 1 \text{ (why?)} \end{cases} \tag{6.32}$$

- Apply (6.32) to (6.31) by setting $\alpha = e^{-j\hat{\omega}}$

$$\begin{aligned} H(e^{j\hat{\omega}}) &= \frac{1}{L}\left(\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}}\right) \\ &= \frac{1}{L}\left(\frac{e^{-j\hat{\omega}L/2}(e^{j\hat{\omega}L/2}-e^{-j\hat{\omega}L/2})}{e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2})}\right) \\ &= \left(\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}\right)e^{-j\hat{\omega}(L-1)/2} \end{aligned} \tag{6.33}$$

- Notice that the phase response is composed of a linear term $e^{-j\hat{\omega}(L-1)/2}$ and $\pm\pi$ due to the sign changes of $\sin(\hat{\omega}L/2)/[L\sin(\hat{\omega}/2)]$
- In MATLAB

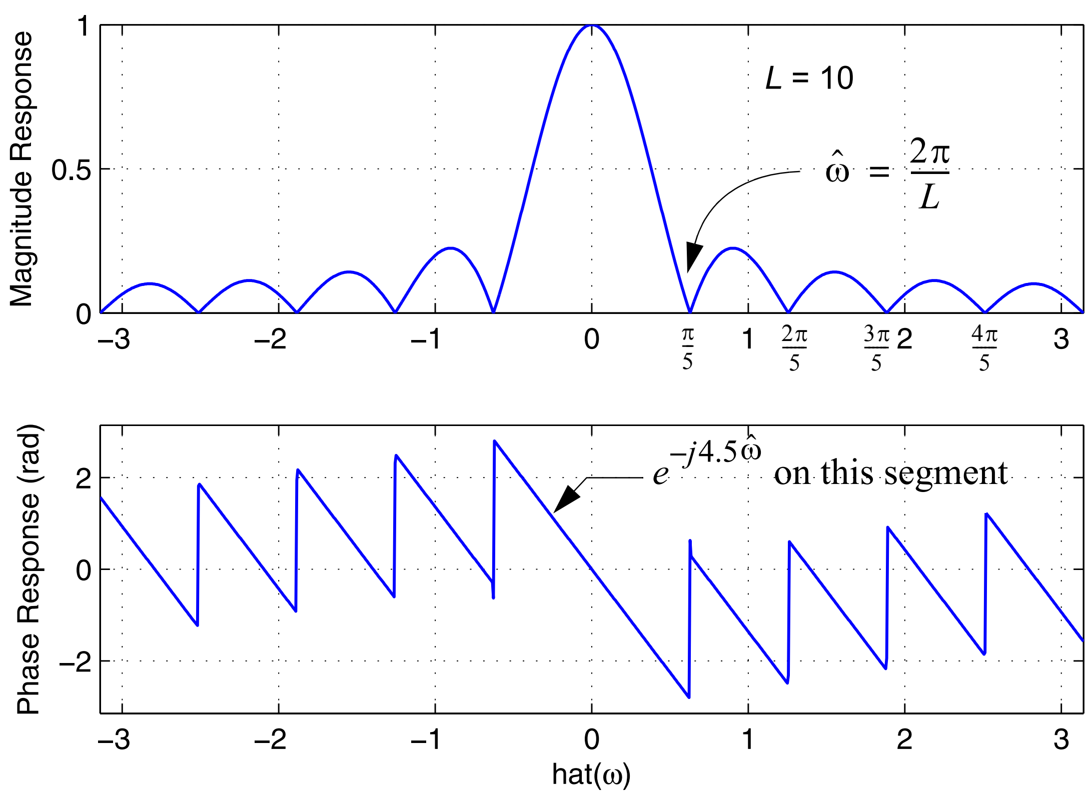
$$D_L(e^{j\hat{\omega}}) = \texttt{diric}(\hat{\omega}, L) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

can be used for analyzing moving average filters

Plotting the Frequency Response

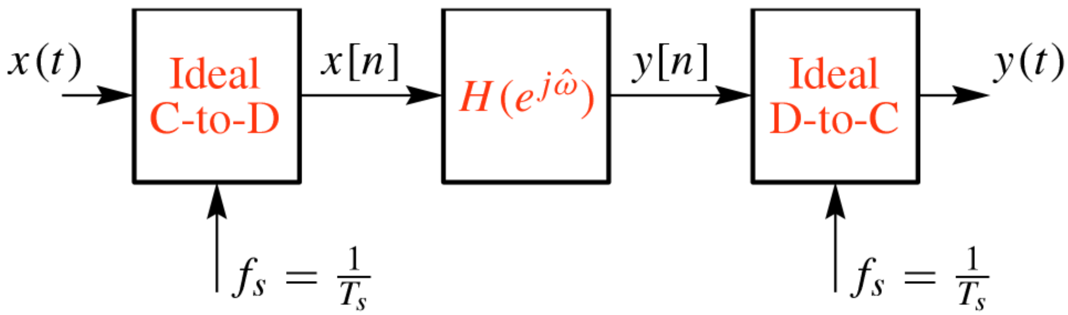
- The frequency response can be plotted most easily using MATLAB's `freqz()` function
- Consider a 10-point moving average

```
>> w = -pi:pi/500:pi;
>> H = freqz(ones(1,10)/10,1,w);
>> subplot(211)
>> plot(w,abs(H))
>> grid; axis([-pi pi 0 1])
>> ylabel('Magnitude Response')
>> subplot(212)
>> plot(w,angle(H))
>> grid; axis([-pi pi -pi pi])
>> ylabel('Phase Response (rad)')
>> xlabel('hat(\omega)')
```



Filtering Sampled Continuous-Time Signals

- Consider the system model as shown below



- Suppose that

$$x(t) = Xe^{j\omega t}, -\infty < t < \infty \tag{6.34}$$

- We know that after sampling we have

$$x[n] = x(nT_s) = Xe^{j\omega nT_s} = Xe^{j\hat{\omega}n} \tag{6.35}$$

- The key relationship to connect the continuous-time and the discrete-time quantities is

$$\boxed{\begin{aligned} \hat{\omega} &= \omega T_s = 2\pi \cdot \frac{f}{f_s} \\ \omega &= \frac{\hat{\omega}}{T_s} = \hat{\omega} f_s \\ f &= \frac{\hat{\omega}}{2\pi T_s} = \frac{\hat{\omega}}{2\pi} \cdot f_s \end{aligned}} \tag{6.36}$$

- As long as the input analog frequency satisfies the sampling theorem, i.e., $|\omega| < \pi/T_s$ or $|f| < f_s/2$, the output of the discrete-time system will be

$$y[n] = H(e^{j\hat{\omega}})Xe^{j\hat{\omega}n} \tag{6.37}$$

- We can write $y[n]$ in terms of the analog frequency variable ω via $\hat{\omega} = \omega T_s$

$$y[n] = H(e^{j\omega T_s})Xe^{j\omega T_s n} \tag{6.38}$$

- At the output of the D-to-C converter we have the reconstructed output

$$\begin{aligned} y(t) &= H(e^{j\omega T_s})Xe^{j\omega t} \\ &= H(e^{j2\pi f/f_s})Xe^{j(2\pi f/f_s)t} \end{aligned} \tag{6.39}$$

where f is the input analog sinusoid frequency (perhaps a better notation would be $f_0 \leftrightarrow \omega_0 \leftrightarrow \hat{\omega}_0$)

<u>Example</u>: Lowpass Averager

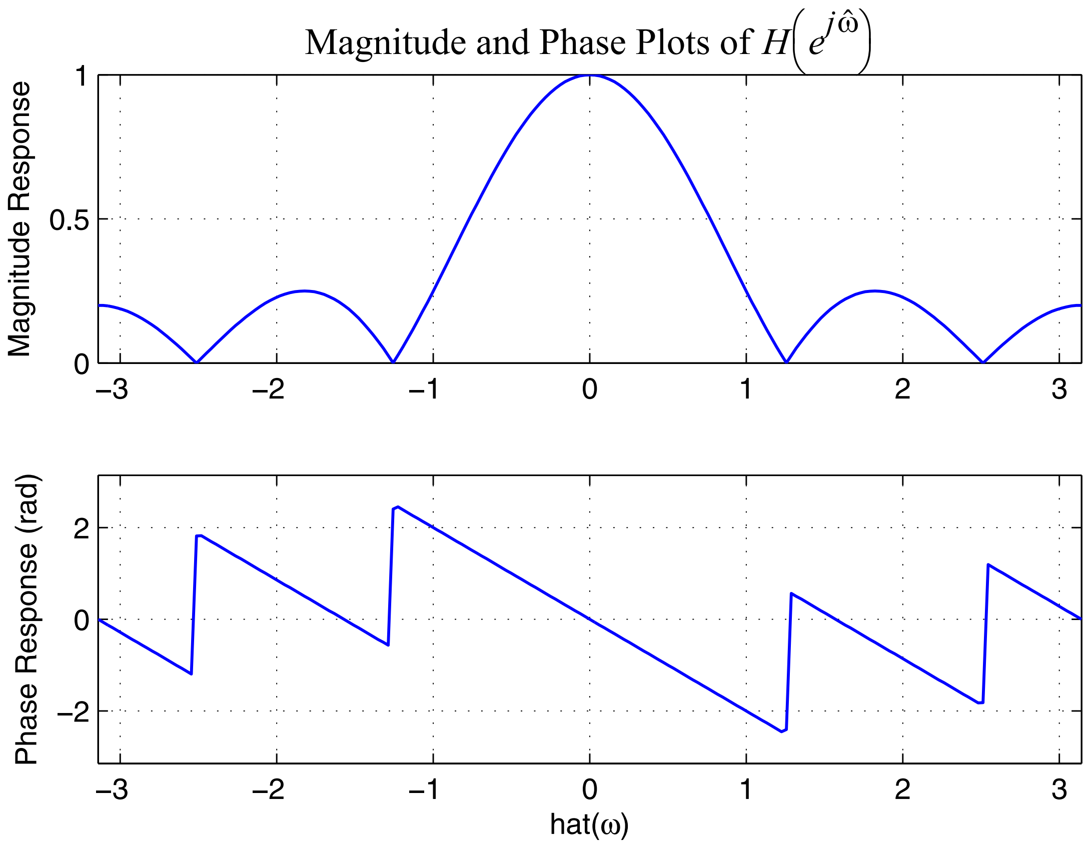
- Consider a 5-point moving average filter wrapped up between a C-to-D and D-to-C system
- We assume a sampling rate of 1000 Hz and an input composed of two sinusoids

$$x(t) = \cos[2\pi(100)t] + 3\cos[2\pi(300)t]$$

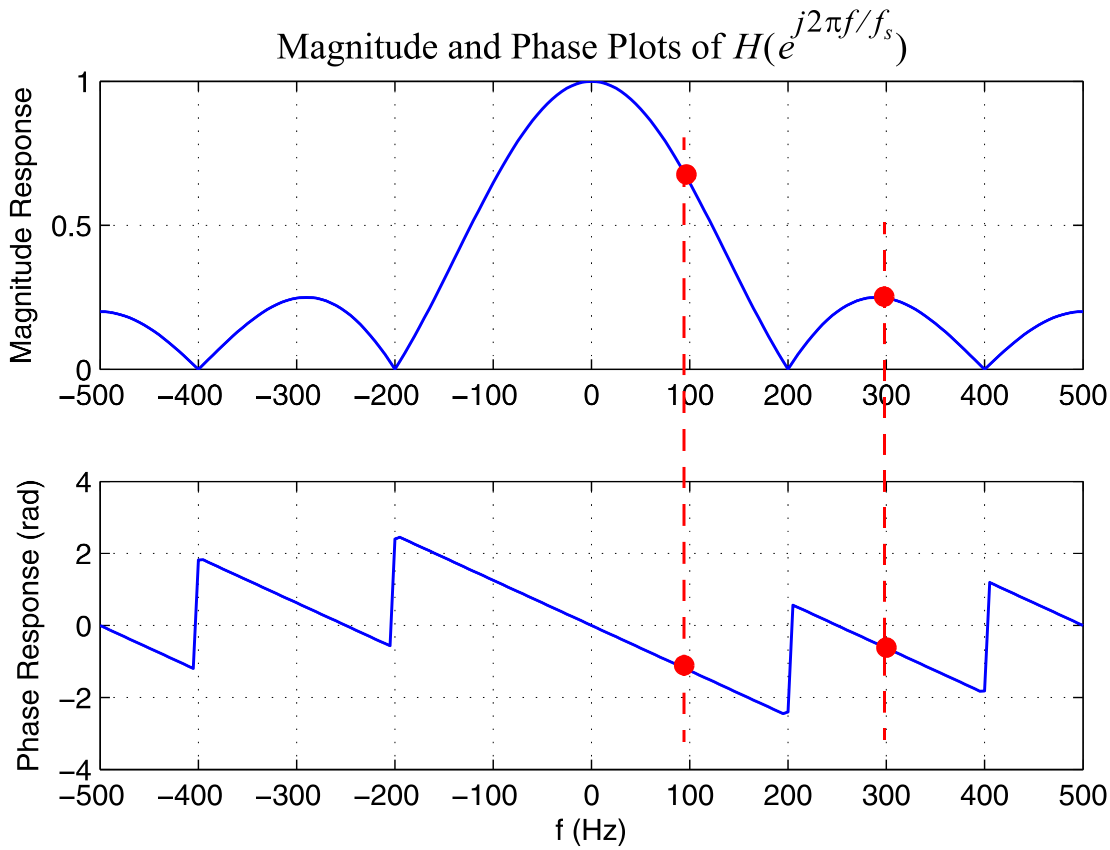
- Find the system frequency response in terms of the analog frequency variable f, and find the steady-state output $y(t)$
- We will use `freqz()` to obtain the frequency response

```
>> w = -pi:pi/100:pi;
>> H = freqz(ones(1,5)/5,1,w);
>> subplot(211)
```

```
>> plot(w,abs(H))
>> axis([-pi pi 0 1]); grid
>> ylabel('Magnitude Response')
>> subplot(212)
>> plot(w,angle(H))
>> axis([-pi pi -pi pi]); grid
>> ylabel('Phase Response (rad)')
>> xlabel('hat(\omega)')
```




```
>> subplot(211)
>> plot(w*1000/(2*pi),abs(H))
>> grid
>> ylabel('Magnitude Response')
>> subplot(212)
>> plot(w*1000/(2*pi),angle(H))
>> grid
>> ylabel('Phase Response (rad)')
>> xlabel('f (Hz)')
```



- The output $y(t)$ will be of the same form as the input $x(t)$, except the sinusoids at 100 and 300 Hz need to have the filter frequency response applied
  - <u>Note</u> 100 Hz and 300 Hz < 1000/2 = 500 Hz (no aliasing)
- To properly apply the filter frequency response we need to convert the analog frequencies to the corresponding discrete-time frequencies

$$\begin{aligned} 100\text{Hz} &\rightarrow 2\pi \cdot \frac{100}{1000} = 2\pi \cdot 0.1 = 0.2\pi \\ 300\text{Hz} &\rightarrow 2\pi \cdot \frac{300}{1000} = 2\pi \cdot 0.3 = 0.6\pi \end{aligned} \tag{6.40}$$

- The frequency response for $L = 5$ in the general moving average filter is

$$H(e^{j\hat{\omega}}) = \frac{\sin(2.5\hat{\omega})}{5\sin(\hat{\omega}/2)}e^{-j2\hat{\omega}} \tag{6.41}$$

- The frequency response at these two frequencies is

$$\begin{aligned} H(e^{j0.2\pi}) &= \frac{\sin(2.5(0.2\pi))}{5\sin((0.2\pi)/2)}e^{-j2(0.2\pi)} = 0.6472e^{-j0.4\pi} \\ H(e^{j0.6\pi}) &= \frac{\sin(2.5(0.6\pi))}{5\sin((0.6\pi)/2)}e^{-j2(0.6\pi)} = \underbrace{-0.2472e^{-j1.2\pi}}_{0.2472e^{-j0.2\pi}} \end{aligned} \tag{6.42}$$

```
>> diric(0.2*pi,5)

ans = 0.6472

>> diric(0.6*pi,5)

ans = -0.2472 % also = 0.2472 at angle +/- pi
```

- The filter output $y[n]$ is

$$\begin{aligned} y[n] &= 0.6472\cos[0.2\pi n - 0.4\pi] \\ &\quad + 0.7416\cos[0.2\pi n - 1.2\pi + \pi] \end{aligned} \tag{6.43}$$

and the D-to-C output is

$$\begin{aligned} y(t) &= 0.6472\cos[2\pi(100)t - 0.4\pi] \\ &\quad + 0.7416\cos[2\pi(300)t - 0.2\pi] \end{aligned} \tag{6.44}$$

Interpretation of Delay

- In the study of the moving average filter, it was noted that
$$H(e^{j\hat{\omega}}) = D_L(e^{j\hat{\omega}})e^{-j\hat{\omega}(L-1)/2}$$
where $D_L(e^{j\hat{\omega}})$ is a purely real function
- The pure delay system
$$y[n] = x[n-n_0]$$
has impulse response
$$h[n] = \delta[n-n_0]$$
and has frequency response
$$H(e^{j\hat{\omega}}) = e^{-jn_0\hat{\omega}}$$
- We also know that the impulse response of two systems in cascade is
$$h[n] = h_1[n]*h_2[n]$$
$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$$
- The L-point moving average filter thus incorporates a delay of $(L-1)/2$ samples
- Viewed as a cascade of subsystems
$$H(e^{j\hat{\omega}}) = D_L(e^{j\hat{\omega}}) \cdot e^{-j\hat{\omega}(L-1)/2} \tag{6.45}$$
  - If L is odd then the delay is an integer
  - If L is even then the delay is an odd half integer